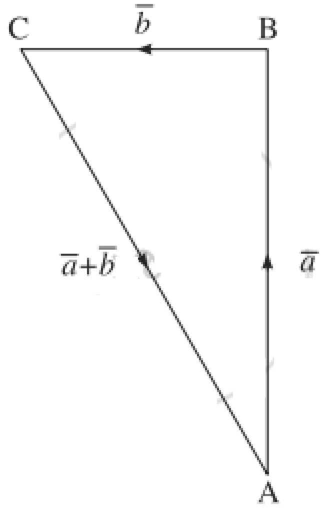
Vectors

EXERCISE 5.1 [PAGES 151 - 152]

Exercise 5.1 | Q 1 | Page 151

The vector $\bar{a}$ is directed due north and |a| = 24. The vector $\bar{b}$ is directed due west and $|\bar{b}| = 7$. Find $|\bar{a} + \bar{b}|$.

Solution:



Let $\overline{AB} = \bar{a}, \overline{BC} = \bar{b}$

Then $\overline{AC} = \overline{AB} + \overline{BC} = \bar{a} + \bar{b}$

Given: $|\bar{a}| = |\overline{AB}| = l(AB) = 24$ and

$|\bar{b}| = |\overline{BC}| = l(BC) = 7$

∵ ∠ABC = 90°

$\therefore [l(AC)]^2 = [l(AB)]^2 + [l(BC)]^2$

$= (24)^2 + (7)^2 = 625$

∴ l(AC) = 25

$\therefore \left|\overline{AC}\right| = 25$

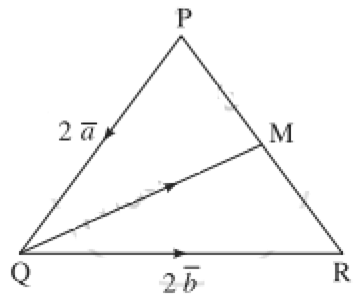
$\therefore \left|\bar{a} + \bar{b}\right| = \left|\overline{AC}\right| = 25$

Exercise 5.1 | Q 2 | Page 151

In the triangle PQR, $\overline{PQ} = \overline{2a}, \overline{QR} = \overline{2b}$. The midpoint of PR is M. Find the following vectors in terms of $\bar{a}$ and $\bar{b}$:

(i) $\overline{PR}$ (ii) $\overline{PM}$ (iii) $\overline{QM}$.

Solution:



$\overline{PQ} = \overline{2a}, \overline{QR} = \overline{2b}$

(i) $\overline{PR} = \overline{PQ} + \overline{QR}$

$= 2\bar{a} + 2\bar{b}$

(ii) $\because$ M is the midpoint of PR

$\therefore \overline{PM} = \frac{1}{2}\overline{PR} = \frac{1}{2}\left[2\bar{a} + 2\bar{b}\right]$

$= \bar{a} + \bar{b}$

(iii) $\overline{RM} = \frac{1}{2}\left(\overline{RP}\right) = -\frac{1}{2}\overline{PR} = -\frac{1}{2}\left(2\bar{a} + 2\bar{b}\right)$

$= -\bar{a} - \bar{b}$

$\therefore \overline{QM} = \overline{QR} + \overline{RM}$

$= 2\bar{b} - \bar{a} - \bar{b}$

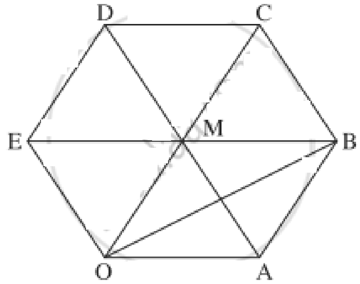
$= \bar{b} - \bar{a}$

[**Note:** point (i) answer in the textbook is incorrect.]

Exercise 5.1 | Q 3 | Page 151

OABCDE is a regular hexagon. The points A and B have position vectors $\bar{a}$ and $\bar{b}$ respectively referred to the origin O. Find, in terms of $\bar{a}$ and $\bar{b}$ the position vectors of C, D and E.

Solution:



Given: $\overline{OA} = \bar{a}, \overline{OB} = \bar{b}$

Let AD, BE, OC meet at M.

Then M bisects AD, BE, OC.

$\overline{AB} = \overline{AO} + \overline{OB}$

$= -\overline{OA} + \overline{OB}$

$= -\bar{a} + \bar{b} = \bar{b} - \bar{a}$

$\because$ OABM is a parallelogram

$\therefore \overline{OM} = \overline{AB} = \bar{b} - \bar{a}$

$\overline{OC} = 2\overline{OM} = 2\left(\bar{b} - \bar{a}\right) = 2\bar{b} - 2\bar{a}$

$\overline{OD} = \overline{OC} + \overline{CD}$

$= \overline{OC} - \overline{DC}$

$= \overline{OC} - \overline{OA}$ ...[∵ OA = DC and OA || DC]

$= 2\bar{b} - 2\bar{a} - \bar{a}$

$= 2\bar{b} - 3\bar{a}$

$\overline{OE} = \overline{OM} + \overline{ME}$

$= (\bar{b} - \bar{a}) - \overline{EM}$

$= \bar{b} - \bar{a} - \bar{a}$ ....[∵ EM = OA and EM || OA]

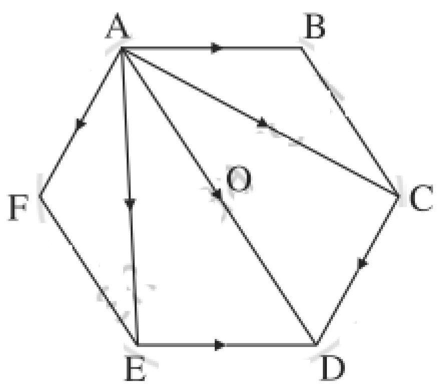
$= \bar{b} - 2\bar{a}$

Hence, the position vectors of C, D and E are $2\bar{b} - 2\bar{a}, 2\bar{b} - 3\bar{a} \text{ and } \bar{b} - 2\bar{a}$ respectively.

Exercise 5.1 | Q 4 | Page 151

ABCDEF is a regular hexagon. Show that $\overline{AB} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF} = 6\overline{AO}$, where O is the centre of the hexagon.

Solution:



ABCDEF is a regular hexagon.

$\therefore \overline{AB} = \overline{ED}$ and $\overline{AF} = \overline{CD}$

$\therefore$ by the triangle law of addition of vectors,

$\overline{AC} + \overline{AF} = \overline{AC} + \overline{CD} = \overline{AD}$

$\overline{AE} + \overline{AB} = \overline{AE} + \overline{ED} = \overline{AD}$

$\therefore$ LHS = $\overline{AB} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF}$

$= \overline{AD} + \left(\overline{AC} + \overline{AF}\right) + \left(\overline{AE} + \overline{AB}\right)$

$= \overline{AD} + \overline{AD} + \overline{AD}$

$= 3\overline{AD} = 3\left(2\overline{AO}\right)$ ....[O is midpoint of AD]

$= 6\overline{AO}$.

Exercise 5.1 | Q 5 | Page 151

Check whether the vectors $2\hat{i} + 2\hat{j} + 3\hat{k}, \ -3\hat{i} + 3\hat{j} + 2\hat{k}$ and $3\hat{i} + 4\hat{k}$ form a triangle or not.

Solution:

Let, if possible, the three vectors form a triangle ABC with $\overline{AB} = 2\hat{i} + 2\hat{j} + 3\hat{k}$,

$\overline{BC} = -3\hat{i} + 3\hat{j} + 2\hat{k}$

$\overline{AC} = 3\hat{i} + 4\hat{k}$

Now, $\overline{AB} + \overline{BC}$

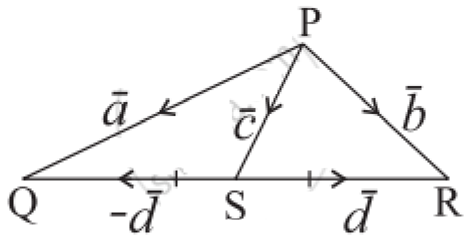
$= \left(2\hat{i} + 2\hat{j} + 3\hat{k}\right) + \left(-3\hat{i} + 3\hat{j} + 2\hat{k}\right)$

= $-\hat{i} + 5\hat{j} + 5\hat{k} \neq 3\hat{i} + 4\hat{k} = \overline{AC}$

Hence, the three vectors do not form a triangle.

Exercise 5.1 | Q 6 | Page 151

In the given figure express $\bar{c}$ and $\bar{d}$ in terms of $\bar{a}$ and $\bar{b}$.



Solution:

$$\overline{PQ} = \overline{PS} + \overline{SQ}$$

$\therefore \bar{a} = \bar{c} - \bar{d}$ ...(1)

$$\overline{PR} = \overline{PS} + \overline{SR}$$

$\therefore \bar{b} = \bar{c} + \bar{d}$ ....(2)

Adding equations (1) and (2), we get

$$\bar{a} + \bar{b} = (\bar{c} - \bar{d}) + (\bar{c} + \bar{d}) = 2\bar{c}$$

$$\therefore \bar{c} = \frac{1}{2}(\bar{a} + \bar{b}) = \frac{1}{2}(\bar{a} + \bar{b})$$

$$= \frac{2\bar{b} - (\bar{a} + \bar{b})}{2}$$

$$= \frac{1}{2}(\bar{b} - \bar{a}) = \frac{1}{2}\bar{b} - \frac{1}{2}\bar{a}$$

Hence, $\bar{c} = \frac{1}{2}\bar{a} + \frac{1}{2}\bar{b}$ and $\bar{d} = \frac{1}{2}\bar{b} - \frac{1}{2}\bar{a}$

Exercise 5.1 | Q 7 | Page 151

Find a vector the direction of $\bar{a} = \hat{i} - 2\hat{j}$ that has magnitude 7 units.

Solution:

$$\bar{a} = \hat{i} - 2\hat{j}$$

$$\therefore |\bar{a}| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$$

Unit vector in direction of $\bar{a}$

$$= \hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{\hat{i} - 2\hat{j}}{\sqrt{5}}$$

$\therefore$ vector of magnitude 7 in the direction of $\bar{a}$ is

$$7\hat{a} = 7\left(\frac{\hat{i} - 2\hat{j}}{\sqrt{5}}\right) = \frac{7}{\sqrt{5}}\hat{i} - \frac{14}{\sqrt{5}}\hat{j}$$

Exercise 5.1 | Q 8 | Page 151

Find the distance of (4, - 2, 6) from each of the following:

(a) The XY-plane

(b) The YZ-plane

(c) The XZ-plane

(d) The X-axis

(e) The Y-axis

(f) The Z-axis.

Solution: Let the point A be (4, - 2, 6).

Then,

(a) The distance of A from XY-plane = |z| = 6

(b) The distance of A from YZ-plane = |x| = 4

(c) The distance of A from ZX-plane = |y| = 2

(d) The distance of A from X-axis

$$= \sqrt{y^2 + z^2} = \sqrt{(-2)^2 + 6^2} = \sqrt{40} = 2\sqrt{10}$$

(e) The distance of A from Y-axis

$$\sqrt{z^2 + x^2} = \sqrt{6^2 + 4^2} = \sqrt{52} = 2\sqrt{13}$$

(f) The distance of A from Z-axis

$$= \sqrt{x^2 + y^2} = \sqrt{\hat{}\ 2 + (-2)^2} = \sqrt{20} = 2\sqrt{5}$$

Exercise 5.1 | Q 9.1 | Page 152

Find the coordinates of the point which is located three units behind the YZ-plane, four units to the right of XZ-plane, and five units above the XY-plane.

Solution: Let the coordinates of the point be (x, y, z). Since the point is located 3 units behind the YZ-plane, 4 units to the right of XZ-plane and 5 units above the XY-plane, x = -3, y = 4 and z = 5

Hence, coordinates of the required point are (3, 4, 5)

Exercise 5.1 | Q 9.2 | Page 152

Find the coordinates of the point which is located in the YZ-plane, one unit to the right of the XZ- plane, and six units above the XY-plane.

Solution: Let the coordinates of the point be (x, y, z). Since the point is located in the YZ plane, x0. Also, the point is one unit to the right of XZ-plane and six units above the XY-plane.
∴ y = 1, z = 6

Hence, coordinates of the required point are (0,1, 6).

Exercise 5.1 | Q 10 | Page 152

Find the area of the traingle with vertices (1, 1, 0), (1, 0, 1) and (0, 1, 1).

Solution:

Let A = (1, 1, 0), B = (1, 0, 1), C = (0, 1, 1)

l(AB) = $\sqrt{(1-1)^2+(1-0)^2+(0-1)^2} = \sqrt{0+1+1} = \sqrt{2}$

l(BC) = $\sqrt{(1-0)^2+(0-1)^2+(1-1)^2} = \sqrt{1+1+0} = \sqrt{1}$

l(CA) = $\sqrt{(0-1)^2+(1-1)^2+(1-0)^2} = \sqrt{1+0+1} = \sqrt{2}$

∴ l(AB) = l(BC) = l(CA)

∴ the triangle is equilateral

∴ its area = $\frac{\sqrt{3}}{4}(\text{side})^2$

$= \frac{\sqrt{3}}{4}\left(\sqrt{2}\right)^2$

$= \frac{\sqrt{3}}{2}$ sq units.

Exercise 5.1 | Q 11 | Page 152

If $\overline{\mathrm{AB}} = 2\hat{\mathrm{i}} - 4\hat{\mathrm{j}} + 7\hat{\mathrm{k}}$ and initial point A(1, 5, 0). Find the terminal point B.

Solution:

Let $\bar{\mathrm{a}}$ and $\bar{\mathrm{b}}$ be the position vectors of A and B.

Given: A(1, 5, 0)

$\therefore \bar{\mathrm{a}} = \hat{\mathrm{i}} + 5\hat{\mathrm{j}}$

Now, $\overline{\mathrm{AB}} = 2\hat{\mathrm{i}} - 4\hat{\mathrm{j}} + 7\hat{\mathrm{k}}$

$\therefore \bar{\mathrm{b}} - \bar{\mathrm{a}} = 2\hat{\mathrm{i}} - 4\hat{\mathrm{j}} + 7\hat{\mathrm{k}}$

$\therefore \bar{\mathrm{b}} = \left(2\hat{\mathrm{i}} - 4\hat{\mathrm{j}} + 7\hat{\mathrm{k}}\right) + \bar{\mathrm{a}}$

$$= \left(2\hat{i} - 4\hat{j} + 7\hat{k}\right) + \left(\hat{i} + 5\hat{j}\right)$$

$$= 3\hat{i} + \hat{j} + 7\hat{k}$$

Hence, the terminal point B(3, 1, 7).

Exercise 5.1 | Q 12.1 | Page 152

Show that the following points are collinear:

A = (3, 2, -4), B = (9, 8, -10), C = (-2, -3, 1)

Solution:

Let $\bar{a}, \bar{b}, \bar{c}$ be the position vectors of the points.

A = (3, 2, - 4), B = (9, 8, - 10) and C = (- 2, - 3, 1) respectively.

Then, $\bar{a} = 3\hat{i} + 2\hat{j} - 4\hat{k},\ \bar{b} = 9\hat{i} + 8\hat{j} - 10\hat{k}, \bar{c} = -2\hat{i} - 3\hat{j} + \hat{k}$

$$\overline{AB} = \bar{b} - \bar{a}$$

$$= \left(9\hat{i} + 8\hat{j} - 10\hat{k}\right) - \left(3\hat{i} + 2\hat{j} - 4\hat{k}\right)$$

$$= 6\hat{i} + 6\hat{j} - 6\hat{k} \quad \text{......(1)}$$

and $\overline{BC} = \bar{c} - \bar{b}$

$$= \left(-2\hat{i} - 3\hat{j} + \hat{k}\right) - \left(9\hat{i} + 8\hat{j} - 10\hat{k}\right)$$

$$= -11\hat{i} - 11\hat{j} + 11\hat{k}$$

$$= 11\left(\hat{i} + \hat{j} - \hat{k}\right)$$

$$= -\frac{11}{6}\left(6\hat{i} + 6\hat{j} - 6\hat{k}\right)$$

$$= -\frac{11}{6}\overline{AB}$$ ....[By (1)]

$\therefore$ $\overline{BC}$ is a non-zero scalar multiple of $\overline{AB}$

$\therefore$ they are parallel to each other.

But they have point B in common.

$\therefore$ $\overline{BC}$ and $\overline{AB}$ are collinear vectors.

Hence, points A, B and C are collinear.

Exercise 5.1 | Q 12.2 | Page 152

Show that the following points are collinear:

P = (4, 5, 2), Q = (3, 2, 4), R = (5, 8, 0).

Solution:

Let $\bar{a}, \bar{b}, \bar{c}$ be position vectors of the points.

P = (4, 5, 2), Q = (3, 2, 4), R = (5, 8, 0) respectively.

Then $\bar{a} = 4\hat{i} + 5\hat{j} + 2\hat{k},\ \bar{b} = 3\hat{i} + 2\hat{j} + 4\hat{k},\ \bar{c} = 5\hat{i} + 8\hat{j} + 0\hat{k}$

$$\overline{AB} = \bar{b} - \bar{a}$$

$$= \left(3\hat{i} + 2\hat{j} + 4\hat{k}\right) - \left(4\hat{i} + 5\hat{j} + 2\hat{k}\right)$$

$= -\hat{i} - 3\hat{j} + 2\hat{k}$ i.e.

$$= -\left(\hat{i} + 3\hat{j} - 2\hat{k}\right)$$ .....(1)

and $\overline{BC} = \bar{c} - \bar{b}$

$= \left(5\hat{i} + 8\hat{j} + 0\hat{k}\right) - \left(3\hat{i} + 2\hat{j} + 4\hat{k}\right)$

$= 2\hat{i} + 6\hat{j} - 4\hat{k}$

$= 2\left(\hat{i} + 3\hat{j} - 2\hat{k}\right)$

$= 2.\overline{AB}$ ....[By(1)]

$\therefore$ $\overline{BC}$ is a non-zero scalar multiple of $\overline{AB}$

$\therefore$ they are parallel to each other.

But they have point B in common.

$\therefore$ $\overline{BC}$ and $\overline{AB}$ are collinear vectors.

Hence, the points A, B, and C are collinear.

Exercise 5.1 | Q 13 | Page 152

If the vectors $2\hat{i} - q\hat{j} + 3\hat{k}$ and $4\hat{i} - 5\hat{j} + 6\hat{k}$ are collinear, find q.

Solution:

The vectors $2\hat{i} - q\hat{j} + 3\hat{k}$ and $4\hat{i} - 5\hat{j} + 6\hat{k}$ are collinear

$\therefore$ the coefficients of $\hat{i}, \hat{j}, \hat{k}$ are proportional

$$\therefore \frac{2}{4} = \frac{-q}{-5} = \frac{3}{6}$$

$$\therefore \frac{q}{5} = \frac{1}{2}$$

$$\therefore q = \frac{5}{2}$$

Exercise 5.1 | Q 14 | Page 152

Are the four points A(1, -1, 1), B(-1, 1, 1), C(1, 1, 1) and D(2, -3, 4) coplanar? Justify your answer.

Solution:

The position vectors $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ of the points A, B, C, D are

$$\bar{a} = \hat{i} - \hat{j} + \hat{k}, \bar{b} = -\hat{i} + \hat{j} + \hat{k},\ \bar{c} = \hat{i} + \hat{j} + \hat{k},\ \bar{d} = 2\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\therefore \overline{AB} = \bar{b} - \bar{a}$$

$$= \left(-\hat{i} + \hat{j} + \hat{k}\right) - \left(\hat{i} - \hat{j} + \hat{k}\right)$$

$$= -2\hat{i} + 2\hat{j}$$

$$\overline{AC} = \bar{c} - \bar{a}$$

$$= \left(\hat{i} + \hat{j} + \hat{k}\right) - \left(\hat{i} - \hat{j} + \hat{k}\right) = 2\hat{j}$$

and $\overline{AD} = \bar{d} - \bar{a} = \left(2\hat{i} - 3\hat{j} + 4\hat{k}\right) - \left(\hat{i} - \hat{j} + \hat{k}\right)$

$$= \hat{i} - 2\hat{j} + 3\hat{k}$$

If A, B, C, D are coplanar, then there exist scalars x, y such that

$$\overline{AB} = x.\overline{AC} + y.\overline{AD}$$

$$\therefore -2\hat{i} + 2\hat{j} = x\left(2\hat{j}\right) + y\left(\hat{i} - 2\hat{j} + 3\hat{k}\right)$$

$$\therefore -2\hat{i} + 2\hat{j} = y\hat{i} + (2x - 2y)\hat{j} + 3y\hat{k}$$

By equality of vectors,

y = - 2 ....(1)

2x - 2y = 2 .....(2)

3y = 0 ....(3)

From (1), y = - 2

From (3), y = 0

This is not possible.

Hence, the points A, B, C, D are not coplanar.

Exercise 5.1 | Q 15 | Page 152

Express $-\hat{i}-3\hat{j}+4\hat{k}$ as the linear combination of the vectors $2\hat{i}+\hat{j}-4\hat{k}, 2\hat{i}-\hat{j}+3\hat{k}$ and $3\hat{i}+\hat{j}-2\hat{k}$.

Solution:

Let $\bar{a}=2\hat{i}+\hat{j}-4\hat{k}$,

$\bar{b}=2\hat{i}-\hat{j}+3\hat{k}$,

$\bar{c}=3\hat{i}+\hat{j}-2\hat{k}$

$\bar{p}=-\hat{i}-3\hat{j}+4\hat{k}$

Suppose $\bar{p}=x\bar{a}+y\bar{b}+z\bar{c}$.

Then, $-\hat{i}-3\hat{j}+4\hat{k}=x\left(2\hat{i}+\hat{j}-4\hat{k}\right)+y\left(2\hat{i}-\hat{j}+3\hat{k}\right)+z\left(3\hat{i}+\hat{j}-2\hat{k}\right)$

$\therefore -\hat{i}-3\hat{j}+4\hat{k}=(2x+2y+3z)\hat{i}+(x-y+z)\hat{j}+(-4x+3y-2z)\hat{k}$

By equality of vectors,

2x + 2y + 3z = -1

x - y + z = - 3

- 4x + 3y - 2z = 4

We have to solve these equations by using Cramer's Rule.

$$D = \begin{vmatrix} 2 & 2 & 3 \\ 1 & -1 & 1 \\ -4 & 3 & -2 \end{vmatrix}$$

= 2(2 - 3) - 2(- 2 + 4) + 3(3 - 4)

= - 2 - 4 - 3

= - 9 ≠ 0

$$D_x = \begin{vmatrix} -1 & 2 & 3 \\ -3 & -1 & 1 \\ 4 & 3 & -2 \end{vmatrix}$$

= - 1(2 - 3) - 2(6 - 4) + 3(- 9 + 4)

= 1 - 4 - 15

= - 18

$$D_y = \begin{vmatrix} 2 & -1 & 3 \\ 1 & -3 & 1 \\ -4 & 4 & -2 \end{vmatrix}$$

= 2(6 - 4) + 1(- 2 + 4) + 3(4 - 12)

= 4 + 2 - 24

= -18

$$D_z = \begin{vmatrix} 2 & 2 & -1 \\ 1 & -1 & -3 \\ -4 & 3 & 4 \end{vmatrix}$$

= 2(- 4 + 9) - 2(4 - 12) - 1(3 - 4)

= 10 + 16 + 1 = 27

∴ x = $\frac{D_x}{D} = \frac{-18}{-9} = 2$

∴ y = $\frac{D_y}{D} = \frac{-18}{-9} = 2$

∴ z = $\frac{D_z}{D} = \frac{27}{-9} = -3$

$\therefore \bar{p} = 2\bar{a} + 2\bar{b} - 3\bar{c}$

EXERCISE 5.2 [PAGE 160]

Exercise 5.2 | Q 1.1 | Page 160

Find the position vector of point R which divides the line joining the points P and Q whose position vectors are $2\hat{i} - \hat{j} + 3\hat{k}$ and $-5\hat{i} + 2\hat{j} - 5\hat{k}$ in the ratio 3 : 2 is internally.

Solution:

It is given that the points P and Q have position vectors $\bar{p} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\bar{q} = -5\hat{i} + 2\hat{j} - 5\hat{k}$ respectively.

If R($\bar{r}$) divides the line segment PQ internally in the ratio 3 : 2, by section formula for internal division,

$$\bar{r} = \frac{3\bar{q} + 2\bar{p}}{3 + 2}$$

$$= \frac{3\left(-5\hat{i} + 2\hat{j} - 5\hat{k}\right) + 2\left(2\hat{i} - \hat{j} + 3\hat{k}\right)}{5}$$

$$= \frac{-11\hat{i} + 4\hat{j} - 9\hat{k}}{5}$$

$$= \frac{1}{5}\left(-11\hat{i} + 4\hat{j} - 9\hat{k}\right)$$

∴ coordinates of R = $\left(-\frac{11}{5}, \frac{4}{5}, -\frac{9}{5}\right)$

Hence, the position vector of R is $\frac{1}{5}\left(-11\hat{i} + 4\hat{j} - 9\hat{k}\right)$ and the coordinates of R are $\left(-\frac{11}{5}, \frac{4}{5}, -\frac{9}{5}\right)$

Exercise 5.2 | Q 1.2 | Page 160

Find the position vector of point R which divides the line joining the points P and Q whose position vectors are $2\hat{i} - \hat{j} + 3\hat{k}$ and $-5\hat{i} + 2\hat{j} - 5\hat{k}$ in the ratio 3 : 2 is externally.

Solution:

It is given that the points P and Q have position vectors $\bar{p} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\bar{q} = -5\hat{i} + 2\hat{j} - 5\hat{k}$ respectively.

If R($\bar{r}$) divides the line segment joining P and Q externally in the ratio 3 : 2, by section formula for external division,

$$\bar{r} = \frac{3\bar{q} - 2\bar{p}}{3 - 2}$$

$$= \frac{3\left(-5\hat{i} + 2\hat{j} - 5\hat{k}\right) - 2\left(2\hat{i} - \hat{j} + 3\hat{k}\right)}{3 - 2}$$

$$= -19\hat{i} + 8\hat{j} - 21\hat{k}$$

∴ coordinates of R = (- 19, 8, -21).

Hence, the position vector of R is $-19\hat{i} + 8\hat{j} - 21\hat{k}$ and coordinates of R are (-19, 8, -21).

Exercise 5.2 | Q 2 | Page 160

Find the position vector of midpoint M joining the points L(7, - 6, 12) and N(5, 4, - 2).

Solution:

The position vectors $\bar{l}$ and $\bar{n}$ of the points L(7, -6, 12) and N(5, 4, -2) are given by

$$\bar{l} = 7\hat{i} - 6\hat{j} + 12\hat{k}, \bar{n} = 5\hat{i} + 4\hat{j} - 2\hat{k}$$

If M($\bar{m}$) is the midpoint of LN, by midpoint formula,

$$\bar{m} = \frac{\bar{l} + \bar{n}}{2}$$

$$= \frac{\left(7\hat{i} - 6\hat{j} + 12\hat{k}\right) + \left(5\hat{i} + 4\hat{j} - 2\hat{k}\right)}{2}$$

$$= \frac{1}{2}\left(12\hat{i} - 2\hat{j} + 10\hat{k}\right) = 6\hat{i} - \hat{j} + 5\hat{k}$$

∴ coordinates of M(6, - 1, 5).

Hence, position vector of M is $6\hat{i} - \hat{j} + 5\hat{k}$ and the coordinates of M are (6, - 1, 5).

Exercise 5.2 | Q 3 | Page 160

If the points A (3, 0, p), B (- 1, q, 3) and C (- 3, 3, 0) are collinear, then find

(i) the ratio in which the point C divides the line segment AB

(ii) the values of p and q.

Solution:

Let $\bar{a}, \bar{b}, \bar{c}$ be the position vectors of A, B and C respectively.

Then $\bar{a} = 3\hat{i} + 0.\hat{j} + p\hat{k}$,

$\bar{b} = -\hat{i} + q\hat{j} + 3\hat{k}$ and

$\bar{c} = -3\hat{i} + 3\hat{j} + 0.\hat{k}$

(i) As the points A, B, C are collinear, suppose the point C divides line segment AB in the ratio λ : 1.

∴ by the section formula,

$$\bar{c} = \frac{\lambda.\bar{b} + 1.\bar{a}}{\lambda + 1}$$

$$\therefore -3\hat{i} + 3\hat{j} + 0.\hat{k}$$

$$= \frac{\lambda\left(-\hat{i} + q\hat{j} + 3\hat{k}\right) + \left(3\hat{i} + 0.\hat{j} + p\hat{k}\right)}{\lambda + 1}$$

$$\therefore (\lambda + 1)\left(-3\hat{i} + 3\hat{j} + 0.\hat{k}\right) = \left(-\lambda\hat{i} + \lambda q\hat{j} + 3\lambda\hat{k}\right) + \left(3\hat{i} + 0.\hat{j} + p\hat{k}\right)$$

$$\therefore -3(\lambda + 1)\hat{i} + 3(\lambda + 1)\hat{j} + 0.\hat{k} = (-\lambda + 3)\hat{i} + \lambda q\hat{j} + (3\lambda + p)\hat{k}$$

By equality of vectors, we have,

$-3(\lambda + 1) = -\lambda + 3$ ....(1)

$3(\lambda + 1) = \lambda q$ .....(2)

$0 = 3\lambda + p$ ....(3)

From equation (1), $-3\lambda - 3 = -\lambda + 3$

∴ $-2\lambda = 6$

∴ $\lambda = -3$

∴ C divides segment AB externally in the ratio 3 : 1.

(ii) Putting $\lambda = -3$ in equation (2), we get

$3(-3 + 1) = -3q$

∴ $-6 = -3q$

∴ q = 2

Also, putting λ = - 3 in equation (3), we get

0 = - 9 + p

∴ p = 9

Hence p = 9 and q = 2.

Exercise 5.2 | Q 4 | Page 160

The position vector of points A and B are $6\bar{a} + 2\bar{b}$ and $\bar{a} - 3\bar{b}$. If the point C divides AB in the ratio 3 : 2, show that the position vector of C is $3\bar{a} - \bar{b}$.

Solution:

Let $\bar{c}$ be the position vector of C.

Since C divides AB in the ratio 3 : 2,

$$\bar{c} = \frac{3(\bar{a} - 3\bar{b}) + 2(6\bar{a} + 2\bar{b})}{3 + 2}$$

$$= \frac{3\bar{a} - 9\bar{b} + 12\bar{a} + 4\bar{b}}{5}$$

$$= \frac{1}{5}(15\bar{a} - 5\bar{b}) = 3\bar{a} - \bar{b}$$

Hence, the position vector of C is $3\bar{a} - \bar{b}$.

Exercise 5.2 | Q 5 | Page 160

Prove that the line segments joining the midpoints of the adjacent sides of a quadrilateral form a parallelogram.

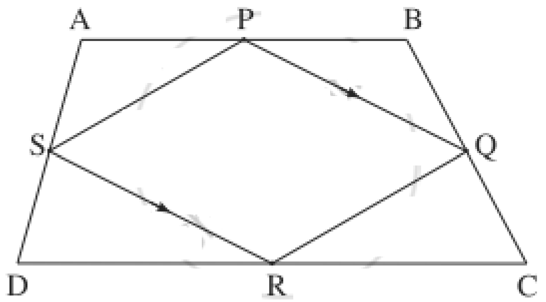
Solution:

Let ABCD be a quadrilateral and P, Q, R, S be the midpoints of the sides AB, BC, CD, and DA respectively.

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{p}, \bar{q}, \bar{r}, \text{ and } \bar{s}$ be the position vectors of the points A, B, C, D, P, Q, R and S respectively.

Since P, Q, R and S are the midpoints of the sides AB, BC, CD and DA respectively,

$$\bar{p} = \frac{\bar{a}+\bar{b}}{2}, \bar{q} = \frac{\bar{b}+\bar{c}}{2}, \bar{r} = \frac{\bar{c}+\bar{d}}{2} \text{ and } \bar{s} = \frac{\bar{d}+\bar{a}}{2}$$



$$\therefore \overline{PQ} = \bar{q} - \bar{p}$$

$$= \left(\frac{\bar{b}+\bar{c}}{2}\right) - \left(\frac{\bar{a}+\bar{b}}{2}\right)$$

$$= \frac{1}{2}(\bar{b}+\bar{c}-\bar{a}-\bar{b}) = \frac{1}{2}(\bar{c}-\bar{a})$$

$$\overline{SR} = \bar{r} - \bar{s}$$

$$= \left(\frac{\bar{c}+\bar{d}}{2}\right) - \left(\frac{\bar{d}+\bar{a}}{2}\right)$$

$$= \frac{1}{2}(\bar{c}+\bar{d}-\bar{d}-\bar{a}) = \frac{1}{2}(\bar{c}-\bar{a})$$

$$\therefore \overline{PQ} = \overline{SR}$$

$\therefore \overline{PQ} \parallel \overline{SR}$

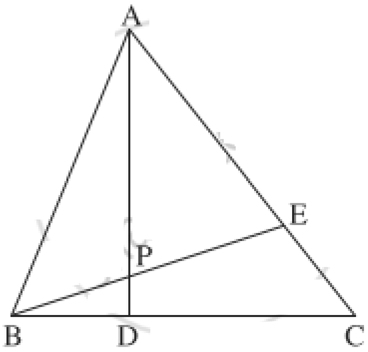
Similarly $\overline{QR} \parallel \overline{PS}$

∴ □PQRS is a parallelogram.

Exercise 5.2 | Q 6 | Page 160

D and E divide sides BC and CA of a triangle ABC in the ratio 2 : 3 each. Find the position vector of the point of intersection of AD and BE and the ratio in which this point divides AD and BE.

Solution:



Let AD and BE intersect at P.

Let A, B, C, D, E, P have position vectos $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{p}$ respectively.

D and E divide segments BC and CA internally in the ratio 2 : 3.

By the section formula for internal division,

$$\bar{d} = \frac{2\bar{c} + 3\bar{b}}{2+3}$$

$\therefore 5\bar{d} = 2\bar{c} + 3\bar{b}$ ....(1)

and $\bar{e} = \dfrac{2\bar{a} + 3\bar{c}}{2+3}$

$\therefore 5\bar{e} = 2\bar{a} + 3\bar{c}$ ....(2)

From (1), $5\bar{d} - 3\bar{b} = 2\bar{c}$ $\therefore 15\bar{d} - 9\bar{b} = 6\bar{c}$

From (2), $5\bar{e} - 2\bar{a} = 3\bar{c}$ $\therefore 10\bar{e} - 4\bar{a} = 6\bar{c}$

Equating both values of $6\bar{c}$, we get

$$15\bar{d} - 9\bar{b} = 10\bar{e} - 4\bar{a}$$

$$\therefore 15\bar{d} + 4\bar{a} = 10\bar{e} + 9\bar{b}$$

$$\therefore \frac{15\bar{d} + 4\bar{a}}{15 + 4} = \frac{10\bar{e} + 9\bar{b}}{10 + 9}$$

LHS is the position vector of the point which divides segment AD internally in the ratio 15 : 4.

RHS is the position vector of the point which divides segment BE internally in the ratio 10 : 9.

But P is the point of intersection of AD and BE.

∴ P divides AD internally in the ratio 15 : 4 and P divides BE internally in the ratio 10 : 9.

Hence, the position vector of the point of interaction of AD and BE is

$$\bar{p} = \frac{15\bar{d} + 4\bar{a}}{19} = \frac{10\bar{e} + 9\bar{b}}{19}$$

and it divides AD internally in the ratio 15 : 4 and BE internally in the ratio 10 : 9.

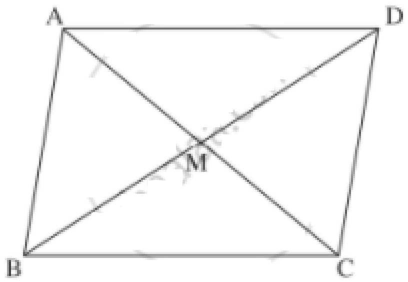
Exercise 5.2 | Q 7 | Page 160

Prove that a quadrilateral is a parallelogram if and only if its diagonals bisect each other.

Solution:

(i) Let $\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$ be respectively the position vectors of the vertices A, B, C and D of the parallelogram ABCD.

Then AB = DC and side AB || side DC.



$\therefore \overline{AB} = \overline{DC}$

$\therefore \bar{b} - \bar{a} = \bar{c} - \bar{d}$

$\therefore \bar{a} + \bar{c} = \bar{b}\,\bar{d}$

$\therefore \dfrac{\bar{a} + \bar{c}}{2} = \dfrac{\bar{b} + \bar{d}}{2}$ ....(1)

The position vectors of the midpoints of the diagonals AC and BD are $\dfrac{\bar{a} + \bar{c}}{2}$ and $\dfrac{\bar{b} + \bar{d}}{2}$.

By (1), they are equal.

∴ the midpoints of the diagonals AC and BD are the same.

This shows that the diagonals AC and BD bisect each other.

(ii) Conversely, suppose that the diagonals AC and BD of □ ABCD bisect each other, i.e. they have the same midpoint.

∴ the position vectors of these midpoints are equal.

$$\therefore \frac{\bar{a} + \bar{c}}{2} = \frac{b + d}{2}$$

$$\therefore \bar{a} + \bar{c} = \bar{b} + \bar{d}$$

$$\therefore \bar{b} - \bar{a} = \bar{c} - \bar{d}$$

$$\therefore \overline{AB} = \overline{DC}$$

$\therefore \overline{AB} \parallel \overline{DC}$ and $\left|\overline{AB}\right| = \left|\overline{DC}\right|$

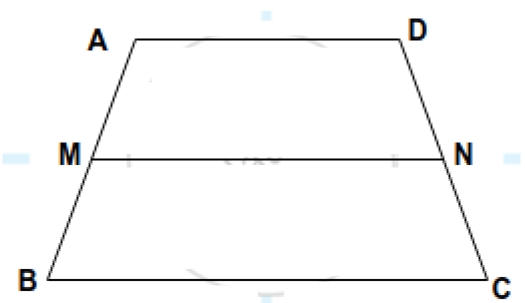
∴ side AB || side DC and AB = DC

∴ $\square ABCD$ is a parallelogram.

Exercise 5.2 | Q 8 | Page 160

Prove that the median of a trapezium is parallel to the parallel sides of the trapezium and its length is half of the sum of the lengths of the parallel sides.

Solution:



Let $\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$ be respectively the position vectors of the vertices A, B, C and D of the trapezium ABCD, with side AD || side BC.

Then the vectors $\overline{AD}$ and $\overline{BC}$ are parallel.

∴ there exists a scalar k,

such that $\overline{AD} = k.\overline{BC}$

$\therefore \overline{AD} + \overline{BC} = k.\overline{BC} + \overline{BC}$

$= (k+1)\overline{BC}$ ....(1)

Let $\overline{m}$ and $\overline{n}$ be the position vectors of the midpoints M and N of the non-parallel sides AB and DC respectively. Then seg MN is the median of the trapezium.

By the midpoint formula,

$\overline{m} = \frac{\bar{a}+\bar{b}}{2}$ and $\bar{n} = \frac{\bar{d}+\bar{c}}{2}$

$\therefore \overline{MN} = \bar{n} - \overline{m}$

$= \left(\frac{\bar{d}+\bar{c}}{2}\right) - \left(\frac{\bar{a}+\bar{b}}{2}\right)$

$= \frac{1}{2}(\bar{d}+\bar{c}-\bar{a}-\bar{b})$

$= \frac{1}{2}[(\bar{d}-\bar{a})+(\bar{c}-\bar{b})]$

$= \frac{\overline{AD}+\overline{BC}}{2}$ .....(2)

$= \frac{(k+1)\overline{BC}}{2}$ ....[By (1)]

Thus $\overline{MN}$ is a scalar multiple of $\overline{BC}$

$\therefore \overline{MN}$ and $\overline{BC}$ are parallel vectors

$\therefore \overline{MN} \parallel \overline{BC}$ where $\overline{BC} \parallel \overline{AD}$

$\therefore$ the median MN is parallel to the parallel sides AD and BC of the trapezium.

Now $\overline{AD}$ and $\overline{BC}$ are collinear

$$\therefore \left|\overline{AD} + \overline{BC}\right| = \left|\overline{AD}\right| + \left|\overline{BC}\right| = AD + BC$$

$\therefore$ from (2), we have

$$\text{Now } \overline{MN} = \frac{\overline{AD} + \overline{BC}}{2}$$

$$\therefore MN = \frac{1}{2}(AD + BC)$$

Exercise 5.2 | Q 9 | Page 160

If two of the vertices of a triangle are A (3, 1, 4) and B(- 4, 5, - 3) and the centroid of the triangle is at G (- 1, 2, 1), then find the coordinates of the third vertex C of the triangle.

Solution:

Let $\bar{a}, \bar{b}, \bar{c}$ and $\bar{g}$ be the position vectors of A, B, C and G respectively.

Then, $\bar{a} = 3\hat{i} + \hat{j} + 4\hat{k}, \bar{b} = -4\hat{i} + 5\hat{j} - 3\hat{k}$ and
$\bar{g} = -\hat{i} + 2\hat{j} + \hat{k}$.

Since G is the centroid of the Δ ABC, by the centroid formula,

$$\bar{g} = \frac{\bar{a} + \bar{b} + \bar{c}}{3}$$

$$\therefore 3\bar{g} = \bar{a} + \bar{b} + \bar{c}$$

$$\therefore 3\left(-\hat{i} + 2\hat{j} + \hat{k}\right) = \left(3\hat{i} + \hat{j} + 4\hat{k}\right) + \left(-4\hat{i} + 5\hat{j} - 3\hat{k}\right) + \bar{c}$$

$\therefore -3\hat{i}+6\hat{j}+3\hat{k}=\left(-\hat{i}+6\hat{j}+\hat{k}\right)+\bar{c}$

$\therefore \bar{c}=\left(-3\hat{i}+6\hat{j}+3\hat{k}\right)-\left(-\hat{i}+6\hat{j}+\hat{k}\right)$

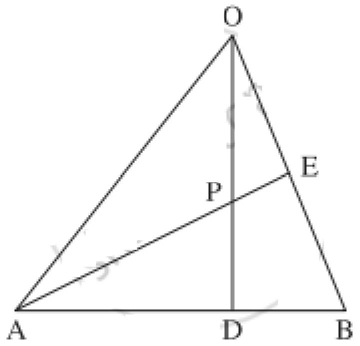
$\therefore \bar{c}=-2\hat{i}+0.\hat{j}+2\hat{k}$

∴ the coordinates of third vertex C are (- 2, 0, 2).

Exercise 5.2 | Q 10 | Page 160

In Δ OAB, E is the midpoint of OB and D is the point on AB such that AD : DB = 2 : 1. If OD and AE intersect at P, then determine the ratio OP : PD using vector methods.

Solution:



Let A, B, D, E, P have position vectors $\bar{a}, \bar{b}, \bar{d}, \bar{e}, \bar{p}$ respectively w.r.t. O.

∵ AD : DB = 2 : 1

∴ D divides AB internally in the ratio 2 : 1.

Using section formula for internal division, we get

$$\bar{d} = \frac{2\bar{b} + \bar{a}}{2+1}$$

$\therefore 3\bar{d} = 2\bar{b} + \bar{a}$ ...(1)

Since E is the midpoint of OB, $\bar{e} = \overline{OE} = \frac{1}{2}\overline{OB} = \frac{\bar{b}}{2}$

$\therefore \bar{b} = 2\bar{e}$ ....(2)

$\therefore$ from (1),

$3\bar{d} = 2(2\bar{e}) + \bar{a}$ ...[By(2)]

$= 4\bar{e} + \bar{a}$

$$\therefore \frac{3\bar{d} + 2.\bar{0}}{3+2} = \frac{4\bar{e} + \bar{a}}{4+1}$$

LHS is the position vector of the point which divides OD internally in the ratio 3 : 2.

RHS is the position vector of the point which divides AE internally in the ratio 4 : 1.

But OD and AE intersect at P

∴ P divides OD internally in the ratio 3 : 2.

Hence, OP : PD = 3 : 2.

Exercise 5.2 | Q 11 | Page 160

If the centroid of a tetrahedron OABC is (1, 2, - 1) where A(a, 2, 3), B(1, b, 2), C(2, 1, c), find the distance of P(a, b, c) from origin.

Solution:

Let G = (1, 2, -1) be the centroid of the tetrahedron OABC.

Let $\bar{a}, \bar{b}, \bar{c}, \bar{g}$ be the position vectors of the points A, B, C, G respectively w.r.t. O.

then $\bar{a} = a\hat{i} + 2\hat{j} + 3\hat{k}$,
$\bar{b} = \hat{i} + b\hat{j} + 2\hat{k}$,
$\bar{c} = 2\hat{i} + \hat{j} + c\hat{k}$,
$\bar{g} = \hat{i} + 2\hat{j} - \hat{k}$

By formula of centroid of a tetrahedron,

$$\bar{g} = \frac{\bar{0} + \bar{a} + \bar{b} + \bar{c}}{4}$$

$$\therefore 4\bar{g} = \bar{a} + \bar{b} + \bar{c}$$

$$\therefore 4\left(\hat{i} + 2\hat{j} - \hat{k}\right) = \left(a\hat{i} + 2\hat{j} + 3\hat{k}\right) + \left(\hat{i} + b\hat{j} + 2\hat{k}\right) + \left(2\hat{i} + \hat{j} + c\hat{k}\right)$$

$$\therefore 4\hat{i} + 8\hat{j} - 4\hat{k} = (a + 1 + 2)\hat{i} + (2 + b + 1)\hat{j} + (3 + 2 + c)\hat{k}$$

$$\therefore 4\hat{i} + 8\hat{j} - 4\hat{k} = (a + 3)\hat{i} + (b + 3)\hat{j} + (c + 5)\hat{k}$$

By equality of vectors

a + 3 = 4, b + 3 = 8, c + 5 = - 4

∴ a = 1, b = 5, c = - 9

∴ P = (a, b, c) = (1, 5, -9)

Distance of P from origin = $\sqrt{1^2 + 5^2 + (-9)^2}$

$= \sqrt{1 + 25 + 81}$

$= \sqrt{107}$ units

Exercise 5.2 | Q 12 | Page 160

Find the centroid of tetrahedron with vertices K(5, - 7, 0), L(1, 5, 3), M(4, - 6, 3), N(6, - 4, 2).

Solution:

Let $\bar{p}, \bar{l}, \bar{m}, \bar{n}$ be the position vectors of the points K, L, M, N respectively w.r.t. the origin O.

Then, $\bar{p} = 5\hat{i} - 7\hat{j} + 0.\hat{k}$

$\bar{l} = \hat{i} + 5\hat{j} + 3\hat{k}$

$\bar{m} = 4\hat{i} - 6\hat{j} + 3\hat{k}$

$\bar{n} = 6\hat{i} - 4\hat{j} + 2\hat{k}$

Let $G(\bar{g})$ be the centroid of the tetrahedron.

Then by centroid formula

$$\bar{g} = \frac{\bar{p} + \bar{l} + \bar{m} + \bar{n}}{4}$$

$$= \frac{1}{4}\left[\left(5\hat{i} - 7\hat{j} + 0.\hat{k}\right) + \left(\hat{i} + 5\hat{j} + 3\hat{k}\right) + \left(4\hat{i} - 6\hat{j} + 3\hat{k}\right) + \left(6\hat{i} - 4\hat{j} + 2\hat{k}\right)\right]$$

$$= \frac{1}{4}\left(16\hat{i} - 12\hat{j} + 8\hat{k}\right)$$

$$= 4\hat{i} - 3\hat{j} + 2\hat{k}$$

Hence, the centroid of the tetrahedron is

G ≡ (4, -3, 2)

EXERCISE 5.3 [PAGES 169 - 170]

Exercise 5.3 | Q 1 | Page 169

Find two unit vectors each of which is perpendicular to both $\bar{u}$ and $\bar{v}$ where $\bar{u} = 2\hat{i} + \hat{j} - 2\hat{k},\ \bar{v} = \hat{i} + 2\hat{j} - 2\hat{k}$.

Solution:

Let $\bar{u} = 2\hat{i} + \hat{j} - 2\hat{k}$,

$\bar{v} = \hat{i} + 2\hat{j} - 2\hat{k}$

Then $\bar{u} \times \bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$

$$= (-2+4)\hat{i} + (-4+2)\hat{j} + (4-1)\hat{k}$$

$$= 2\hat{i} - 2\hat{j} + 3\hat{k}$$

$$\therefore |\bar{u} \times \bar{v}| = \sqrt{(2)^2 + (-2)^2 + (3)^2}$$

$$= \sqrt{4+4+9}$$

$$= \sqrt{17}$$

$$= \pm\frac{\bar{u} \times \bar{v}}{|\bar{u} \times \bar{v}|} = \pm\frac{2\hat{i} - 2\hat{j} + 3\hat{k}}{\sqrt{17}}$$

$$= \pm\left(\frac{2}{\sqrt{17}}\hat{i} - \frac{2}{\sqrt{17}}\hat{j} + \frac{3}{\sqrt{17}}\hat{k}\right)$$

Exercise 5.3 | Q 2 | Page 169

If $\bar{a}$ and $\bar{b}$ are two vectors perpendicular to each other, prove that $(\bar{a}+\bar{b})^2 = (\bar{a}-\bar{b})^2$

Solution:

$\bar{a}$ and $\bar{b}$ are perpendicular to each other.

$\therefore \bar{a}.\bar{b} = \bar{b}.\bar{a} = 0$ ...(1)

LHS = $(\bar{a}+\bar{b})^2$

$$= (\bar{a}+\bar{b}).(\bar{a}+\bar{b})$$

$$= \bar{a}.(\bar{a}+\bar{b}) + \bar{b}(\bar{a}+\bar{b})$$

$$= \bar{a}.\bar{a} + \bar{a}.\bar{b} + \bar{b}.\bar{a} + \bar{b}.\bar{b}$$

$= \bar{a}.\bar{a} + 0 + 0 + \bar{b}.\bar{b}$ ....[By (1)]

$= |\bar{a}|^2 + |\bar{b}|^2$

RHS = $(\bar{a} - \bar{b})^2$

$= (\bar{a} - \bar{b}).(\bar{a} - \bar{b})$

$= \bar{a}.(\bar{a} - \bar{b}) + \bar{b}(\bar{a} - \bar{b})$

$= \bar{a}.\bar{a} - \bar{a}.\bar{b} - \bar{b}.\bar{a} + \bar{b}.\bar{b}$

$= \bar{a}.\bar{a} + \bar{b}.\bar{b}$ ...[By(1)]

$= |\bar{a}|^2 + |\bar{b}|^2$

∴ LHS = RHS

Hence, $(\bar{a} + \bar{b})^2 = (\bar{a} - \bar{b})^2$

Exercise 5.3 | Q 3 | Page 169

Find the values of c so that for all real x, the vectors $xc\hat{i} - 6\hat{j} + 3\hat{k}$ and $x\hat{i} + 2\hat{j} + 2cx\hat{k}$ make an obtuse angle.

Solution:

Let $\bar{a} = xc\hat{i} - 6\hat{j} + 3\hat{k}$ and $\bar{b} = x\hat{i} + 2\hat{j} + 2cx\hat{k}$

Consider $\bar{a}.\bar{b} = \left(xc\hat{i} - 6\hat{j} + 3\hat{k}\right).\left(x\hat{i} + 2\hat{j} + 2cx\hat{k}\right)$

$= (xc)(x) + (-6)(2) + (3)(2cx)$

$= cx^2 - 12 + 6cx$

$= cx^2 + 6cx - 12$

If the angle between $\bar{a}$ and $\bar{b}$ is obtuse, $\bar{a}.\bar{b} < 0$

$\therefore cx^2 + 6cx - 12 < 0$

$\therefore cx^2 + 6cx < 12$

$\therefore c(x^2 + 6x) < 12$

$\therefore c < \frac{12}{x^2 + 6x}$

$\therefore c < \frac{12}{(x^2 + 6x + 9) - 9} = \frac{12}{(x + 3)^2 - 9}$

$\therefore c < \min\left\{\frac{12}{(x + 3)^2 - 9}\right\}$

Now, $\frac{12}{(x + 3)^2 - 9}$ is minimum if $(x + 3)^2 - 9$ is maximum

i.e. $(x + 3)^2 - 9 = \infty - 9 = \infty$

$\therefore c < \min\left\{\frac{12}{\infty}\right\} = 0$

$\therefore c < 0.$

Hence, the angle between $\bar{a}$ and $\bar{b}$ is obtuse if $c < 0$.

Exercise 5.3 | Q 4 | Page 169

Show that the sum of the length of projections of $p\hat{i} + q\hat{j} + r\hat{k}$ on the coordinate axes, where p = 2, q = 3 and r = 4 is 9.

Solution:

Let $\bar{a} = p\hat{i} + q\hat{j} + r\hat{k}$

Projection of $\bar{a}$ on X-axis

$$= \frac{\bar{a}.\,\hat{i}}{\left|\hat{i}\right|} = \frac{\left(p\hat{i} + q\hat{i} + r\hat{k}\right).\,\hat{i}}{1} = p = 2$$

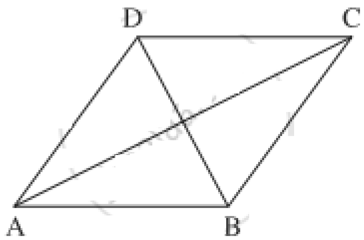
Similarly, projections of $\bar{a}$ on Y- and Z-axes are 3 and 4 respectively.

$\therefore$ sum of these projections = 2 + 3 + 4 = 9.

Exercise 5.3 | Q 5 | Page 169

Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.

Solution:



Let ABCD be a quadrilateral in which

$\left|\overline{AB}\right| = \left|\overline{BC}\right| = \left|\overline{CD}\right| = \left|\overline{DA}\right|$ ....(1)

and AB || DC and AD || BC

$\therefore\ \overline{AB} = \overline{DC}$ and $\overline{AD} = \overline{BC}$ ...(2)

Now, $\overline{AC} = \overline{AB} + \overline{BC}$

and $\overline{BD} = \overline{BA} + \overline{AD} = -\overline{AB} + \overline{BC}$ ...[By(2)]

$= \overline{BC} - \overline{AB}$

$\therefore \overline{AC}.\overline{BD} = \left(\overline{AB} + \overline{BC}\right).\left(\overline{BC} - \overline{AB}\right)$

$= \overline{AB}.\left(\overline{BC} - \overline{AB}\right) + \overline{BC}.\left(\overline{BC} - \overline{AB}\right)$

$= \overline{AB}.\overline{BC} - \overline{AB}.\overline{AB} + \overline{BC}.\overline{BC} - \overline{BC}.\overline{AB}$

$= \left|\overline{BC}\right|^2 - \left|\overline{AB}\right|^2 \quad .... \left[\because \overline{AB}.\overline{BC} = \overline{BC}.\overline{AB}\right]$

= 0 ...[By(1)]

$\because \overline{AC}, \overline{BD}$ are non-zero vectors

$\therefore \overline{AC}$ is perpendicular to $\overline{BD}$

Hence, the diagonals are perpendicular.

Exercise 5.3 | Q 6.1 | Page 169

Determine where $\bar{a}$ and $\bar{b}$ are orthogonal, parallel or neithe:

$\bar{a} = -9\hat{i} + 6\hat{j} + 15\hat{k}, \bar{b} = 6\hat{i} - 4\hat{j} - 10\hat{k}$

Solution:

$\bar{a} = -9\hat{i} + 6\hat{j} + 15\hat{k} = -3\left(3\hat{i} - 2\hat{j} - 5\hat{k}\right)$

$= -\frac{3}{2}\left(6\hat{i} - 4\hat{j} - 19\hat{k}\right)$

$\therefore \bar{a} = -\frac{3}{2}\bar{b}$

i.e. $\bar{a}$ is a non-zero scalar multiple of $\bar{b}$

Hence, $\bar{a}$ is parallel to $\bar{b}$

Exercise 5.3 | Q 6.2 | Page 169

Determine where $\bar{a}$ and $\bar{b}$ are orthogonal, parallel or neithe:

$\bar{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\bar{b} = 5\hat{i} - 2\hat{j} + 4\hat{k}$

Solution:

$\bar{a}.\bar{b} = \left(2\hat{i} + 3\hat{j} - \hat{k}\right).\left(5\hat{i} - 2\hat{j} + 4\hat{k}\right)$

$= (2)(5) + (3)(-2) + (-1)(4)$

= 10 - 6 - 4 = 0

Since, $\bar{a}, \bar{b}$ are non-zero vectors and $\bar{a}.\bar{b} = 0$

$\bar{a}$ is orthogonal to $\bar{b}$

Exercise 5.3 | Q 6.3 | Page 169

Determine where $\bar{a}$ and $\bar{b}$ are orthogonal, parallel or neithe:

$\bar{a} = -\frac{3}{5}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{3}\hat{k}$, $\bar{b} = 5\hat{i} + 4\hat{j} + 3\hat{k}$

Solution:

$\bar{a}.\bar{b} = \left(-\frac{3}{5}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{3}\hat{k}\right).\left(5\hat{i} + 4\hat{j} + 3\hat{k}\right)$

$= \left(-\frac{3}{5}\right)(5) + \left(\frac{1}{2}\right)(4) + \left(\frac{1}{3}\right)(3)$

= -3 + 2 + 1

= 0

Since, $\bar{a}, \bar{b}$ are non-zero vectors and $\bar{a}.\bar{b} = 0$

$\bar{a}$ is orthogonal to $\bar{b}$

Exercise 5.3 | Q 6.4 | Page 169

Determine where $\bar{a}$ and $\bar{b}$ are orthogonal, parallel or neithe:

$\bar{a} = 4\hat{i} - \hat{j} + 6\hat{k}$, $\bar{b} = 5\hat{i} - 2\hat{j} + 4\hat{k}$

Solution:

$$\bar{a}.\bar{b} = \left(4\hat{i} - \hat{j} + 6\hat{k}\right).\left(5\hat{i} - 2\hat{j} + 4\hat{k}\right)$$

$$= (4)(5) + (-1)(-2) + (6)(4)$$

= 20 + 2 + 24

= 46 ≠ 0

$\therefore$ $\bar{a}$ is not orthogonal to $\bar{b}$

It is clear that $\bar{a}$ is not a scalar multiple of $\bar{b}$

$\bar{a}$ is not parallel to $\bar{b}$

Hence, $\bar{a}$ is neither parallel nor orthogonal to $\bar{b}$.

Exercise 5.3 | Q 7 | Page 169

Find the angle P of the triangle whose vertices are P(0, - 1, - 2), Q(3, 1, 4) and R(5, 7, 1).

Solution:

The position vectors $\bar{p}, \bar{q},$ and $\bar{r}$ of the points P(0, - 1, - 2), Q(3, 1, 4) and R(5, 7, 1) are

$\bar{p} = -\hat{j} - 2\hat{k}$,

$\bar{q} = 3\hat{i} + \hat{j} + 4\hat{k}$,

$\bar{r} = 5\hat{i} + 7\hat{j} + \hat{k}$

$$\therefore \overline{PQ} = \bar{q} - \bar{p}$$

$$= \left(3\hat{i} + \hat{j} + 4\hat{k}\right) - \left(-\hat{j} - 2\hat{k}\right)$$

$$= 3\hat{i} + 2\hat{j} + 6\hat{k}$$

and $\overline{PR} = \bar{r} - \bar{p}$

$= \left(5\hat{i} + 7\hat{j} + \hat{k}\right) - \left(-\hat{j} - 2\hat{k}\right)$

$= 5\hat{i} + 8\hat{j} + 3\hat{k}$

$= \overline{PQ}.\overline{PR} = \left(3\hat{i} + 2\hat{j} + 6\hat{k}\right).\left(5\hat{i} + 8\hat{j} + 3\hat{k}\right)$

$= (3)(5) + (2)(8) + (6)(3)$

= 15 + 16 + 18 = 49

$|\overline{PQ}| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{9 + 4 + 36} = \sqrt{49} = 7$

$|\overline{PR}| = \sqrt{5^2 + 8^2 + 3^2} = \sqrt{25 + 64 + 9} = \sqrt{98} = 7\sqrt{2}$

Using the formula for angle between two vectors,

$\cos P = \dfrac{\overline{PQ}.\overline{PR}}{|\overline{PQ}||\overline{PR}|}$

$= \dfrac{49}{7 \times 7\sqrt{2}} = \dfrac{1}{\sqrt{2}} = \cos 45^\circ$

∴ P = 45°.

Exercise 5.3 | Q 8.1 | Page 169

If $\bar{p}, \bar{q}$ and $\bar{r}$ are unit vectors, find $\bar{p}.\bar{q}$.

Solution:

Let the triangle be denoted by ABC, where $\overline{AB} = \bar{p}$, $\overline{AC} = \bar{q}$ and $\overline{BC} = \bar{r}$

∵ $\bar{p}, \bar{q}, \bar{r}$ are unit vectors.

∴ l(AB) = l(BC) = l(CA) = 1

∴ the triangle is equilateral

∴ ∠A = ∠B = ∠C = 60°

Using the formula for angle between two vectors,

$$\cos A = \frac{\bar{p}.\bar{q}}{|\bar{p}|.|\bar{q}|}$$

$$\therefore \cos 60° = \frac{\bar{p}.\bar{q}}{1 \times 1}$$

$$\therefore \frac{1}{2} = \bar{p}.\bar{q}$$

$$\therefore \bar{p}.\bar{q} = \frac{1}{2}$$

Exercise 5.3 | Q 8.2 | Page 169

If $\bar{p}, \bar{q}$ and $\bar{r}$ are unit vectors, find $\bar{p}.\bar{r}$.

Solution:

Let the triangle be denoted by ABC, where $\overline{AB} = \bar{p}$, $\overline{AC} = \bar{q}$ and $\overline{BC} = \bar{r}$

$\because \bar{p}, \bar{q}, \bar{r}$ are unit vectors.

$\therefore$ l(AB) = l(BC) = l(CA) = 1

$\therefore$ the triangle is equilateral

$\therefore \angle A = \angle B = \angle C = 60°$

Using the formula for angle between two vectors,

$$\cos B = \frac{\bar{p}.\bar{r}}{|\bar{p}|.|\bar{r}|}$$

we get $\bar{p}.\bar{r} = \frac{1}{2}$

$\therefore \cos 60° = \dfrac{\bar{p}.\bar{r}}{1 \times 1}$

$\therefore \dfrac{1}{2} = \bar{p}.\bar{r}$

$\therefore \bar{p}.\bar{r} = \dfrac{1}{2}$

Exercise 5.3 | Q 9 | Page 169

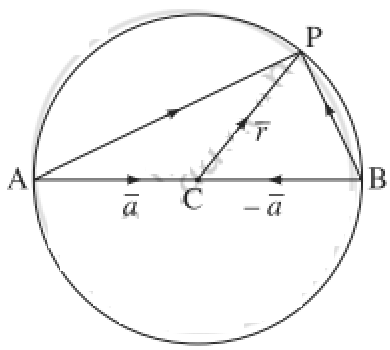
Prove by vector method, that the angle subtended on semicircle is a right angle.

Solution: Let seg AB be a diameter of a circle with centre C and P be any point on the circle other than A and B.

Then ∠APB is an angle subtended on a semicircle.

Let $\overline{AC} = \overline{CB} = \bar{a}$ and $\overline{CP} = \bar{r}$

Then $|\bar{a}| = |\bar{r}|$ ....(1)



$\overline{AP} = \overline{AC} + \overline{CP} = \bar{a} + \bar{r} = \bar{r} + \bar{a}$

$\overline{BP} = \overline{BC} + \overline{CP} = -\overline{CB} + \overline{CP} = -\bar{a} + \bar{r}$

$\therefore \overline{AP}.\overline{BP} = (\bar{r} + \bar{a}).(\bar{r} - \bar{a})$

$= \bar{r}.\bar{r} - \bar{r}.\bar{a} + \bar{a}.\bar{r} - \bar{a}.\bar{a}$

$= |\bar{r}|^2 - |\bar{a}|^2 = 0$ ....$(\because \bar{r}.\bar{a} = \bar{a}.\bar{r})$

$\therefore \overline{AP} \perp \overline{BP}$

∴ ∠APB is a right angle.

Hence, the angle subtended on a semicircle is the right angle.

Exercise 5.3 | Q 10 | Page 169

If a vector has direction angles 45° and 60°, find the third direction angle.

Solution: Let α = 45°, β = 60°

We have to find γ.

∵ $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

∴ $\cos^2 45° + \cos^2 60° + \cos^2 r = 1$

$$\therefore \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right) + \cos^2\gamma = 1$$

$$\therefore \cos^2\gamma = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\therefore \cos\gamma = \pm\frac{1}{2}$$

$$\therefore \cos\gamma = \frac{1}{2} \text{ or } \cos\gamma = -\frac{1}{2}$$

$$\therefore \cos\gamma = \cos\frac{\pi}{3} \text{ or } \cos\gamma = -\cos\frac{\pi}{3}$$

$$\therefore \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$$

$$\therefore \gamma = \frac{\pi}{3} \text{ or } \gamma = \frac{2\pi}{3}$$

Hence, the third direction angle is $\frac{\pi}{3}$ or $\frac{2\pi}{3}$

[**Note:** Answer in the textbook is incomplete.]

Exercise 5.3 | Q 11 | Page 169

If a line makes angles 90°, 135°, 45° with the X-, Y- and Z-axes respectively, then find its direction cosines.

Solution: Let l, m, n be the direction cosines of the line.

Then l = cos α, m = cos β, n = cos γ

Here, α = 90°, β = 135°, γ = 45°

∴ l = cos 90° = 0

m = cos 135° = cos (180° - 45°) = - cos 45°

$= -\frac{1}{\sqrt{2}}$ and n = cos 45° = $\frac{1}{\sqrt{2}}$

∴ the direction cosines of the line are 0, $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

Exercise 5.3 | Q 12 | Page 170

If a line has the direction ratios 4, - 12, 18, then find its direction cosines.

Solution: The direction ratios of the line are a = 4, b = - 12, c = 18.

Let l, m, n be the direction cosines of the line.

Then l = $\frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{4}{\sqrt{4^2+(-12)^2+(18)^2}}$

$= \frac{4}{\sqrt{16+144+324}} = \frac{4}{22} = \frac{2}{11}$

m = $\frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{-12}{\sqrt{4^2+(-12)^2+(18)^2}}$

$= \frac{-12}{\sqrt{16+144+324}} = \frac{-12}{22} = \frac{-6}{11}$

and n = $\frac{c}{\sqrt{a^2+b^2+c^2}}$

$= \frac{18}{\sqrt{4^2+(-12)^2+(18)^2}}$

$= \frac{18}{\sqrt{16+144+324}} = \frac{18}{22} = \frac{9}{11}$

Hence, the direction cosines of the line are $\frac{2}{11}, \frac{-6}{11}, \frac{9}{11}$.

Exercise 5.3 | Q 13 | Page 170

The direction ratios of $\overline{\mathrm{AB}}$ are - 2, 2, 1. If A ≡ (4, 1, 5) and l(AB) = 6 units, find B.

Solution:

The direction ratio of $\overline{\mathrm{AB}}$ are -2, 2, 1.

∴ the direction cosines of $\overline{\mathrm{AB}}$ are

l = $\frac{-2}{\sqrt{(-2)^2+2^2+1^2}} = \frac{-2}{3}$,

m = $\frac{2}{\sqrt{(-2)^2+2^2+1^2}} = \frac{2}{3}$,

n = $\frac{1}{\sqrt{(-2)^2+2^2+1^2}} = \frac{1}{3}$.

i.e. l = $\frac{-2}{3}$, m = $\frac{2}{3}$, n = $\frac{1}{3}$

The coordinates of the points which are at a distance of d units from the point (x_1, y_1, z_1) are given by $(x_1 \pm ld, y_1 \pm md, z_1 \pm nd)$

Here, $x_1 = 4$, $y_1 = 1$, $z_1 = 5$, $d = 6$, $l = -\frac{2}{3}$, $m = \frac{2}{3}$, $n = \frac{1}{3}$

$\therefore$ the coordinates of the required points are

$$\left(4 \pm \left(-\frac{2}{3}\right)6, 1 \pm \frac{2}{3}(6), 5 \pm \frac{1}{3}(6)\right)$$

i.e. (4 - 4, 1 + 4, 5 + 2) and (4 + 4, 1 - 4, 5 - 2)

i.e. (0, 5, 7) and (8, - 3, 3).

Exercise 5.3 | Q 14 | Page 170

Find the angle between the lines whose direction cosines l, m, n satisfy the equations 5l + m + 3n = 0 and 5mn - 2nl + 6lm = 0.

Solution: Given, 5l + m + 3n = 0 ...(1)

and 5mn - 2nl + 6lm = 0 ....(2)

From (1), m = - (5l + 3n)

Putting the value of m in equation (2), we get,

-5(5l + 3n)n - 2nl - 6l(5l + 3n) = 0

$\therefore - 25ln - 15n^2 - 2nl - 30l^2 - 18ln = 0$

$\therefore - 30l^2 - 45ln - 15n^2 = 0$

$\therefore 2l^2 + 3ln + n^2 = 0$

$\therefore 2l^2 + 2ln + ln + n^2 = 0$

$\therefore 2l(l + n) + n(l + n) = 0$

$\therefore (l + n)(2l + n) = 0$

$\therefore l + n = 0$ or $2l + n = 0$

$\therefore l = - n$ or $n = - 2l$

Now, m = - (5l + 3n), therefore, if l = - n,

m = - (5l + 3n), therefore, if l = - n

m = - (- 5n + 3n) = 2n

$\therefore$ - l = $\frac{m}{2} = n$

$\therefore \frac{l}{-1} = \frac{m}{2} = \frac{n}{1}$

$\therefore$ the direction ratios of the first line are

a_1 = - 1, b_1 = 2, c_1 = 1

If n = - 2l, m = -(5l - 6l) = l

$\therefore$ l = m = $\frac{n}{-2}$

$\therefore \frac{l}{1} = \frac{m}{1} = \frac{n}{-2}$

$\therefore$ the direction ratios of the second line are

a_2 = 1, b_2 = 1, c_2 = - 2

Let θ be the angle between the lines.

Then cos θ = $\left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}.\sqrt{a_2^2 + b^2\_2 + c_2^2}}\right|$

$= \left|\frac{(-1)(1) + 2(1) + 1(-2)}{\sqrt{(-1)^2 + 2^2 + 1^2}.\sqrt{1^2 + 1^2 + (-2)^2}}\right|$

$= \left|\frac{-1 + 2 - 2}{\sqrt{6}.\sqrt{6}}\right|$

$= \left|\frac{-1}{6}\right| = \frac{1}{6}$

$\therefore \theta = \cos^{-1}\left(\frac{1}{6}\right)$

[**Note:** Answer in the textbook is incorrect.]

EXERCISE 5.4 [PAGES 178 - 179]

Exercise 5.4 | Q 1 | Page 178

If $\bar{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\bar{b} = \hat{i} - 4\hat{j} + 2\hat{k}$, find $\left(\bar{a} \times \bar{b}\right) \times \left(\bar{a} - \bar{b}\right)$

Solution:

Given: $\bar{a} = 2\hat{i} + 3\hat{j} - \hat{k}$,

$\bar{b} = \hat{i} - 4\hat{j} + 2\hat{k}$

$\therefore \bar{a} + \bar{b} = \left(2\hat{i} + 3\hat{j} - \hat{k}\right) + \left(\hat{i} - 4\hat{j} + 2\hat{k}\right)$

$= 3\hat{i} - \hat{j} + \hat{k}$

and $\bar{a} - \bar{b} = \left(2\hat{i} + 3\hat{j} - \hat{k}\right) - \left(\hat{i} - 4\hat{j} + 2\hat{k}\right)$

$= \hat{i} + 7\hat{j} - 3\hat{k}$

$$\therefore \left(\bar{a} + \bar{b}\right) \times \left(\bar{a} - \bar{b}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ 1 & 7 & -3 \end{vmatrix}$$

$= (3 - 7)\hat{i} - (-9 - 1)\hat{j} + (21 + 1)\hat{k}$

$= -4\hat{i} + 10\hat{j} + 22\hat{k}$.

Exercise 5.4 | Q 2 | Page 178

Find a unit vector perpendicular to the vectors $\hat{j} + 2\hat{k}$ and $\hat{i} + \hat{j}$.

Solution:

Let $\bar{a} = \hat{j} + 2\hat{k}$, $\bar{b} = \hat{i} + \hat{j}$

Then $\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix}$

$= (0 - 2)\hat{i} - (0 - 2)\hat{j} + (0 - 1)\hat{k}$

$= -2\hat{i} + 2\hat{j} - \hat{k}$

$\therefore \left|\bar{a} \times \bar{b}\right| = \sqrt{(-2)^2 + 2^2 + (-1)^2}$

$= \sqrt{4 + 4 + 1} = \sqrt{9} = 3$

Unit vector perpendicular to both $\bar{a}$ and $\bar{b}$

$= \pm \dfrac{\bar{a} \times \bar{b}}{\left|\bar{a} \times \bar{b}\right|} = \pm \left(\dfrac{-2\hat{i} + 2\hat{j} - \hat{k}}{3} \right)$

$= \pm \left(-\dfrac{2}{3}\hat{i} + \dfrac{2}{3}\hat{j} - \dfrac{1}{3}\hat{k} \right)$

[**Note:** Answer in the textbook is incorrect.]

Exercise 5.4 | Q 3 | Page 178

If $\bar{a}.\bar{b} = \sqrt{3}$ and $\bar{a} \times \bar{b} = 2\hat{i} + \hat{j} + 2\hat{k}$, find the angle between $\bar{a}$ and $\bar{b}$.

Solution:

Let θ be the angle between $\bar{a}$ and $\bar{b}$

$\because \bar{a} \times \bar{b} = 2\hat{i} + \hat{j} + 2\hat{k}$

$\therefore \left|\bar{a} \times \bar{b}\right| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{4 + 1 + 4} = 3$

$\therefore |\bar{a}|\left|\bar{b}\right|$ sin θ = 3 ...(1)

$\therefore \bar{a}.\bar{b} = \sqrt{3}$

$\therefore |\bar{a}|\left|\bar{b}\right| \cos\theta = \sqrt{3}$ ....(2)

∴ Dividing (1) by (2), we get

$$\frac{|\bar{a}|\left|\bar{b}\right| \sin\theta}{|\bar{a}|\left|\bar{b}\right| \cos\theta} = \frac{3}{\sqrt{3}}$$

∴ tan θ = $\sqrt{3} = \tan 60^\circ$

∴ θ = 60°

Exercise 5.4 | Q 4 | Page 178

If $\bar{a} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\bar{b} = \hat{i} - 2\hat{j} + \hat{k}$, find a vector of magnitude 5 perpendicular to both $\bar{a}$ and $\bar{b}$.

Solution:

Given: $\bar{a} = 2\hat{i} + \hat{j} - 3\hat{k}$ and

$\bar{b} = \hat{i} - 2\hat{j} + \hat{k}$

$$\therefore \bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -3 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= (1-6)\hat{i} - (2+3)\hat{j} + (-4-1)\hat{k}$$

$$= -5\hat{i} - 5\hat{j} - 5\hat{k}$$

$$\therefore \left|\bar{a} \times \bar{b}\right| = \sqrt{(-5)^2 + (-5)^2 + (-5)^2}$$

$$= \sqrt{25 + 25 + 25} = \sqrt{75} = 5\sqrt{3}$$

$\therefore$ unit vectors perpendicular to both the vectors $\bar{a}$ and $\bar{b}$

$$= \frac{\pm\left(\bar{a} \times \bar{b}\right)}{\left|\bar{a} \times \bar{b}\right|}$$

Given: $\bar{a} = 2\hat{i} + \hat{j} - 3\hat{k}$ and

$\bar{b} = \hat{i} - 2\hat{j} + \hat{k}$

$$\therefore \bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -3 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= (1-6)\hat{i} - (2+3)\hat{j} + (-4-1)\hat{k}$$

$$= -5\hat{i} - 5\hat{j} - 5\hat{k}$$

$$\therefore \left|\bar{a} \times \bar{b}\right| = \sqrt{(-5)^2 + (-5)^2 + (-5)^2}$$

$$= \sqrt{25 + 25 + 25} = \sqrt{75} = 5\sqrt{3}$$

$\therefore$ unit vectors perpendicular to both the vectors $\bar{a}$ and $\bar{b}$

$$= \frac{\pm(\bar{a} \times \bar{b})}{|\bar{a} \times \bar{b}|}$$

$$= \frac{\pm\left(-5\hat{i} - 5\hat{j} - 5\hat{k}\right)}{5\sqrt{3}}$$

$$= \frac{\pm\left(\hat{i} + \hat{j} + \hat{k}\right)}{\sqrt{3}}$$

$\therefore$ required vectors of magnitude 5 units

$$= \pm\frac{5}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right).$$

[**Note:** Answer in the textbook is incorrect.]

Exercise 5.4 | Q 5.1 | Page 178

Find $\bar{u}.\bar{v}$ if $|\bar{u}| = 2, |\bar{v}| = 5, |\bar{u} \times \bar{v}| = 8$

Solution:

Let θ be the angle between $\bar{u}$ and $\bar{v}$

Then $|\bar{u} \times \bar{v}| = 8$ gives

$|\bar{u}||\bar{v}| \sin \theta = 8$

$\therefore 2 \times 5 \times \sin \theta = 8$

$\therefore \sin \theta = \frac{4}{5}$

$\cos \theta = \pm\sqrt{1 - \sin^2\theta}$ ...[$\because 0 \le \theta \le$]

$$= -\sqrt{1-\left(\frac{4}{5}\right)^2}$$

$$= \pm\sqrt{1-\frac{16}{25}}$$

$$= -\sqrt{\frac{9}{25}} = \pm\frac{3}{5}$$

Now, $\bar{u}.\bar{v} = |\bar{u}||\bar{v}|$ cos θ

$$\therefore \bar{u}.\bar{v} = 2 \times 5 \times \left(\pm\frac{3}{5}\right) = \pm 6$$

Exercise 5.4 | Q 5.2 | Page 178

Find $|\bar{u} \times \bar{v}|$ if $|\bar{u}| = 10, |\bar{v}| = 2, \bar{u}.\bar{v} = 12$

Solution:

Let θ be the angle between $\bar{u}$ and $\bar{v}$

Then $\bar{u}.\bar{v} = 12$ gives

$|\bar{u}||\bar{v}|$ cos θ = 12

∴ 10 × 2 × cos θ = 12

∴ cos θ = $\frac{3}{5}$ where $0 \leq \theta \leq \frac{\pi}{2}$

sin θ = $\sqrt{1-\cos^2\theta}$

$$= \sqrt{1-\left(\frac{3}{5}\right)^2}$$

$$= \sqrt{1 - \frac{9}{25}}$$

$$= \sqrt{\frac{16}{25}} = \frac{4}{5}$$

Now, $|\bar{u} \times \bar{v}| = |\bar{u}||\bar{v}| \sin\theta$

$$\therefore \bar{u}.\bar{v} = 10 \times 2 \times \left(\frac{4}{5}\right) = 16$$

[**Note:** Answer in the textbook is incorrect.]

Exercise 5.4 | Q 6 | Page 178

Prove that $2(\bar{a} - \bar{b}) \times 2(\bar{a} + \bar{b}) = 8(\bar{a} \times \bar{b})$

Solution:

LHS = $2(\bar{a} - \bar{b}) \times 2(\bar{a} + \bar{b})$

$= 4[(\bar{a} - \bar{b}) \times (\bar{a} + \bar{b})]$

$= 4[\bar{a} \times (\bar{a} + \bar{b}) - \bar{b} \times (\bar{a} + \bar{b})]$

$= 4(\bar{a} \times \bar{a} + \bar{a} \times \bar{b} - \bar{b} \times \bar{a} - \bar{b} - \bar{b})$

$= 8(\bar{a} \times \bar{b})$ ....$[\because \bar{a} \times \bar{a} = \bar{b} \times \bar{b} = \bar{0} \text{ and} - (\bar{b} - \bar{a}) = \bar{a} \times \bar{b}]$

= RHS

$\therefore 2(\bar{a} - \bar{b}) \times 2(\bar{a} + \bar{b}) = 8(\bar{a} \times \bar{b})$

Exercise 5.4 | Q 7 | Page 178

If $\bar{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\bar{b} = 4\hat{i} - 3\hat{j} + \hat{k}$, $\bar{c} = \hat{i} - \hat{j} + 2\hat{k}$ verify that $\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$

Solution:

Given: $\bar{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\bar{b} = 4\hat{i} - 3\hat{j} + \hat{k}$, $\bar{c} = \hat{i} - \hat{j} + 2\hat{k}$

$\therefore \bar{b} + \bar{c} = \left(4\hat{i} - 3\hat{j} + \hat{k}\right) + \left(\hat{i} - \hat{j} + 2\hat{k}\right)$

$= 5\hat{i} - 4\hat{j} + 3\hat{k}$

and $\bar{a} \times \left(\bar{b} + \bar{c}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 4 & -3 & 1 \end{vmatrix}$

$= (-6 + 12)\hat{i} - (3 - 15)\hat{j} + (-4 + 10)\hat{k}$

$= 6\hat{i} + 12\hat{j} + 6\hat{k}$ ...(1)

Also, $\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 4 & -3 & 1 \end{vmatrix}$

$= (-2 + 9)\hat{i} - (1 - 12)\hat{j} + (-3 + 8)\hat{k}$

$= 7\hat{i} + 11\hat{j} + 5\hat{k}$

and $\bar{a} \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 1 & -1 & 2 \end{vmatrix}$

$= (-4 + 3)\hat{i} - (2 - 3)\hat{j} + (-1 + 2)\hat{k}$

$= -\hat{i} + \hat{j} + \hat{k}$

$\therefore \bar{a} \times \bar{b} + \bar{a} \times \bar{c} = \left(7\hat{i} + 11\hat{j} + 5\hat{k}\right) + \left(-\hat{i} + \hat{j} + \hat{k}\right)$

$= 6\hat{i} + 12\hat{j} + 6\hat{k}$ .....(2)

From (1) and (2), we get

$\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$

Exercise 5.4 | Q 8 | Page 178

Find the area of the parallelogram whose adjacent sides are $\bar{a} = 2\hat{i} - 2\hat{j} + \hat{k}$ and $\bar{b} = \hat{i} - 3\hat{j} - 3\hat{k}$

Solution:

Given: $\bar{a} = 2\hat{i} - 2\hat{j} + \hat{k}$ and $\bar{b} = \hat{i} - 3\hat{j} - 3\hat{k}$

$$\therefore \bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & -3 & -3 \end{vmatrix}$$

$$= (6 + 3)\hat{i} - (-6 - 1)\hat{j} + (-6 + 2)\hat{k}$$

$$= 9\hat{i} + 7\hat{j} - 4\hat{k}$$

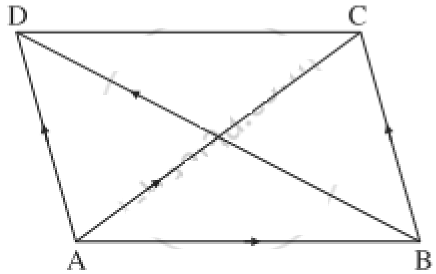
$$|\bar{a} \times \bar{b}| = \sqrt{9^2 + 7^2 + (-4)^2} = \sqrt{81 + 49 + 16} = \sqrt{16}$$

Area of the parallelogram whose adjacent sides are $\bar{a}$ and $\bar{b} = \sqrt{146}$ sq units.

Exercise 5.4 | Q 9 | Page 178

Show that vector area of a parallelogram ABCD is $\frac{1}{2}\left(\overline{AC} \times \overline{BD}\right)$ where AC and BD are its diagonals.

Solution:



Let ABCD be a parallelogram.

Then $\overline{\mathrm{AC}} = \overline{\mathrm{AB}} + \overline{\mathrm{BC}}$ and

$\overline{\mathrm{BD}} = \overline{\mathrm{BA}} + \overline{\mathrm{AD}} = -\overline{\mathrm{AB}} + \overline{\mathrm{BC}} \ldots. \left[\because \overline{\mathrm{BC}} = \overline{\mathrm{AD}}\right]$

$= \overline{\mathrm{BC}} - \overline{\mathrm{AB}}$

$\therefore \overline{\mathrm{AC}} \times \overline{\mathrm{BD}} = \left(\overline{\mathrm{AB}} + \overline{\mathrm{BC}}\right) \times \left(\overline{\mathrm{BC}} - \overline{\mathrm{AB}}\right)$

$= \overline{\mathrm{AB}} \times \left(\overline{\mathrm{BC}} - \overline{\mathrm{AB}}\right) + \overline{\mathrm{BC}} \times \left(\overline{\mathrm{BC}} - \overline{\mathrm{AB}}\right)$

$= \overline{\mathrm{AB}} \times \overline{\mathrm{BC}} - \overline{\mathrm{AB}} \times \overline{\mathrm{AB}} + \overline{\mathrm{BC}} \times \overline{\mathrm{BC}} - \overline{\mathrm{BC}} \times \overline{\mathrm{AB}}$

$= \overline{\mathrm{AB}} \times \overline{\mathrm{BC}} + \overline{\mathrm{AB}} \times \overline{\mathrm{BC}}$

$\ldots. \left[\overline{\mathrm{AB}} \times \overline{\mathrm{AB}} = \overline{\mathrm{BC}} \times \overline{\mathrm{BC}} = \bar{0} \text{ and } -\overline{\mathrm{BC}} \times \overline{\mathrm{AB}} = \overline{\mathrm{AB}} \times \overline{\mathrm{BC}}\right]$

$\therefore \overline{\mathrm{AC}} \times \overline{\mathrm{BD}} = 2\left(\overline{\mathrm{AB}} \times \overline{\mathrm{BC}}\right)$

= 2 (vector area of parallelogram ABCD)

$\therefore$ vector area of parallelogram ABCD $= \frac{1}{2}\left(\overline{\mathrm{AC}} \times \overline{\mathrm{BD}}\right)$

Exercise 5.4 | Q 10 | Page 179

Find the area of parallelogram whose diagonals are determined by the vectors $\bar{a} = 3\hat{i} - \hat{j} - 2\hat{k}$ and $\bar{b} = -\hat{i} + 3\hat{j} - 3\hat{k}$.

Solution:

Given: $\bar{a} = 3\hat{i} - \hat{j} - 2\hat{k}$, $\bar{b} = -\hat{i} + 3\hat{j} - 3\hat{k}$.

$$\therefore \bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & -2 \\ -1 & 3 & -3 \end{vmatrix}$$

$= (3+6)\hat{i} - (-9-2)\hat{j} + (9-1)\hat{k}$

$= 9\hat{i} + 11\hat{j} + 8\hat{k}$

and $|\bar{a} \times \bar{b}| = \sqrt{9^2 + 11^2 + 8^2} = \sqrt{81 + 121 + 64} = \sqrt{266}$

Area of the parallelogram having diagonals $\bar{a}$ and $\bar{b} = \frac{1}{2}|\bar{a} \times \bar{b}| = \frac{1}{2}\sqrt{266}$ sq units.

Exercise 5.4 | Q 11 | Page 179

If $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are four distinct vectors such that $\bar{a} \times \bar{b} = \bar{c} \times \bar{d}$ and $\bar{a} \times \bar{c} = \bar{b} \times \bar{d}$ prove that $\bar{a} - \bar{d}$ is parallel to $\bar{b} - \bar{c}$.

Solution:

$\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are four distinct vectors.

$\therefore \bar{a} \neq \bar{b} \neq \bar{c} \neq \bar{d}$

$\therefore \bar{a} - \bar{d} \neq \bar{0}$ and $\bar{b} - \bar{c} \neq \bar{0}$ ....(1)

Now, $\bar{a} \times \bar{b} = \bar{c} \times \bar{d}$ ...(2)

and $\bar{a} \times \bar{c} = \bar{b} \times \bar{d}$ ...(3)

Subtracting (3) from (2), we get

$\bar{a} \times \bar{b} - \bar{a} \times \bar{c} = \bar{c} \times \bar{d} - \bar{b} \times \bar{d}$

$\therefore \bar{a} \times (\bar{b} - \bar{c}) = (\bar{c} - \bar{b}) \times \bar{d} = -(\bar{b} - \bar{c}) \times \bar{d} = \bar{d} \times (\bar{b} - \bar{c})$

$\therefore \bar{a} \times (\bar{b} - \bar{c}) - \bar{d} \times (\bar{b} - \bar{c}) = \bar{0}$

$\therefore (\bar{a} - \bar{d}) \times (\bar{b} - \bar{c}) = \bar{0}$

$\therefore \bar{a} - \bar{d}$ and $\bar{b} - \bar{c}$ are parallel to each other. ...[By (1)]

Exercise 5.4 | Q 12 | Page 179

If $\bar{a} = \hat{i} + \hat{j} + \hat{k}$ and $\bar{c} = \hat{j} - \hat{k}$, find $\bar{a}$ vector $\bar{b}$ satisfying $\bar{a} \times \bar{b} = \bar{c}$ and $\bar{a}.\bar{b} = 3$

Solution:

Given: $\bar{a} = \hat{i} + \hat{j} + \hat{k},\ \bar{c} = \hat{j} - \hat{k}$

Let $\bar{b} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $\bar{a}.\bar{b} = 3$ gives

$$\left(\hat{i} + \hat{j} + \hat{k}\right).\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = 3$$

∴ (1)(x) + (1)(y) + (1)(z) = 3

Also, x + y + z = 3 ...(1)

Also, $\bar{c} = \bar{a} \times \bar{b}$

$$\therefore \hat{j} - \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix}$$

$$= (z - y)\hat{i} - (z - x)\hat{j} + (y - x)\hat{k}$$

$$= (z - y)\hat{i} + (x - z)\hat{j} + (y - x)\hat{k}$$

By equality of vectors,

z - y = 0 ...(2)

x - z = 1 .....(3)

y - x = - 1 ...(4)

From (2), y = z.

From (3), x = 1 + z

Substituting these values of x and y in (1), we get

1 + z + z + z = 3

∴ z = $\frac{2}{3}$

∴ y = z = $\frac{2}{3}$

∴ x = 1 + z = $1 + \frac{2}{3} = \frac{5}{3}$

$\therefore \bar{b} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

i.e. $\bar{b} = \frac{1}{3}\left(5\hat{i} + 2\hat{j} + 2\hat{k}\right)$

Exercise 5.4 | Q 13 | Page 179

Find $\bar{a}$ if $\bar{a} \times \hat{i} + 2\bar{a} - 5\hat{j} = \bar{0}$

Solution:

Let $\bar{a} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $\bar{a} \times \hat{i} = \left(x\hat{i} + y\hat{j} + z\hat{k}\right) \times \hat{i}$

$= x\left(\hat{i} \times \hat{i}\right) + y\left(\hat{j} \times \hat{i}\right) + z\left(\hat{k} \times \hat{i}\right)$

$= z\hat{j} - y\hat{k} \quad .... \left[\because \hat{i} \times \hat{i} = \hat{0}, \hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{i} = \hat{j}\right]$

It is given that

$\bar{a} \times \hat{i} + 2\bar{a} - 5\hat{j} = \bar{0}$

$$\therefore z\hat{j} - y\hat{k} + 2\left(x\hat{i} + y\hat{j} + z\hat{k}\right) - 5\hat{j} = 0$$

$$\therefore z\hat{j} - y\hat{k} + 2x\hat{i} + 2y\hat{j} + 2z\hat{k} - 5\hat{j} = \bar{0}$$

$$\therefore 2x\hat{i} + (2y + z - 5)\hat{j} + (2z - y)\hat{k} = \bar{0}$$

By equality of vectors

2x = 0 i.e. x = 0

2y + z - 5 = 0 ....(1)

2z - y = 0 ....(2)

From (2), y = 2z

Substituting y = 2z in (1), we get

4z + z = 5

∴ z = 1

∴ y = 2z = 2(1) = 2

∴ x = 0, y = 2, z = 1

$$\therefore \bar{a} = 2\hat{j} + \hat{k}$$

Exercise 5.4 | Q 14 | Page 179

If $\left|\bar{a}.\bar{b}\right| = \left|\bar{a} \times \bar{b}\right|$ and $\bar{a}.\bar{b} < 0$, then find the angle between $\bar{a}$ and $\bar{b}$.

Solution:

Let θ be the angle between $\bar{a}$ and $\bar{b}$

Then $\left|\bar{a}.\bar{b}\right| = \left|\bar{a} \times \bar{b}\right|$ gives

$$|ab\cos\theta| = |ab\sin\theta|$$

∴ - ab cos θ = ab sin θ

∴ - 1 = tan θ

∴ tan θ = - tan $\frac{\pi}{4}$ = tan $\left(\pi - \frac{\pi}{4}\right)$

$\therefore \tan\theta = \tan\dfrac{3\pi}{4}$

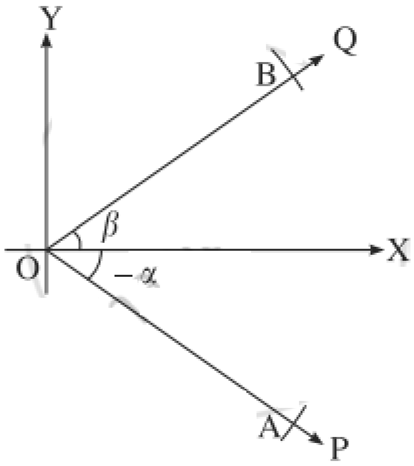
$\therefore \theta = \dfrac{3\pi}{4}$

Hence, the angle between $\bar{a}$ and $\bar{b}$ is $\dfrac{3\pi}{4}$.

Exercise 5.4 | Q 15 | Page 179

Prove, by vector method, that sin (α + β) = sin α . cos β + cos α . sin β

Solution:



Let ∠XOP and ∠XOQ be in standard position and m∠XOP = - α ,m∠XOQ = β

Take a point A on ray OP and a point B on ray OQ such that OA = OB = 1.

Since cos (- α) = cos α

and sin (- α) = - sin α,

A is (cos (- α), sin (- α)),

i.e. (cos α, - sin α)

B is (cos β, sin β)

$\therefore \overline{\mathrm{OA}} = (\cos\alpha)\overline{\mathrm{i}} - (\sin\alpha).\,\overline{\mathrm{j}} + 0.\,\overline{\mathrm{k}}$

$\overline{\mathrm{OB}} = (\cos\beta)\overline{\mathrm{i}} - (\sin\beta).\,\overline{\mathrm{j}} + 0.\,\overline{\mathrm{k}}$

$$\therefore \overline{\mathrm{OA}} \times \overline{\mathrm{OB}} = \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ \cos\alpha & -\sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \end{vmatrix}$$

= (cos α sin β + sin α cos β)$\overline{\mathrm{k}}$ ....(1)

The angle between $\overline{\mathrm{OA}}$ and $\overline{\mathrm{OB}}$ is α + β.

Also, $\overline{\mathrm{OA}}$,bar"OB"` lie in the XY-plane.

∴ the unit vector perpendicular to $\overline{\mathrm{OA}}$ and $\overline{\mathrm{OB}}$ is $\overline{\mathrm{k}}$.

$\therefore \overline{\mathrm{OA}} \times \overline{\mathrm{OB}} = [\mathrm{OA.\,OB}\sin(\alpha+\beta)]\overline{\mathrm{k}}$

= sin (α + β) . $\overline{\mathrm{k}}$ ...(2)

∴ from (1) and (2),

sin (α + β) = sin α cos β + cos α sin β

Exercise 5.4 | Q 16.1 | Page 179

Find the direction ratios of a vector perpendicular to the two lines whose direction ratios are - 2, 1, - 1 and - 3, - 4, 1

Solution: Let a, b, c be the direction ratios of the vector which is perpendicular to the two lines whose direction ratios are -2, 1, -1 and -3, -4, 1

∴ - 2a + b - c = 0 and - 3a - 4b + c = 0

$$\therefore \frac{\mathrm{a}}{\begin{vmatrix} 1 & -1 \\ -4 & 1 \end{vmatrix}} = \frac{\mathrm{b}}{\begin{vmatrix} -1 & -2 \\ 1 & -2 \end{vmatrix}} = \frac{\mathrm{c}}{\begin{vmatrix} -2 & 1 \\ -3 & -4 \end{vmatrix}}$$

$$\therefore \frac{\mathrm{a}}{1-4} = \frac{\mathrm{b}}{3+2} = \frac{\mathrm{c}}{8+3}$$

$$\therefore \frac{\mathrm{a}}{-3} = \frac{\mathrm{b}}{5} = \frac{\mathrm{c}}{11}$$

∴ the required direction ratios are - 3, 5, 11

Alternative Method:

Let $\bar{a}$ and $\bar{b}$ be the vectors along the lines whose direction ratios are -2, 1, -1 and -3, -4, 1 respectively.

Then $\bar{a} = -2\hat{i} + \hat{j} - \hat{k}$ and $\bar{b} = -3\hat{i} - 4\hat{j} + \hat{k}$

The vector perpendicular to both $\bar{a}$ and $\bar{b}$ is given by

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & -1 \\ -3 & -4 & 1 \end{vmatrix}$$

$$= (1 - 4)\hat{i} - (-2 - 3)\hat{j} + (8 + 3)\hat{k}$$

$$= -3\hat{i} + 5\hat{j} + 11\hat{k}$$

Hence, the required direction ratios are - 3, 5, 11.

Exercise 5.4 | Q 16.2 | Page 179

Find the direction ratios of a vector perpendicular to the two lines whose direction ratios are 1, 3, 2 and - 1, 1, 2

Solution:

Let $\bar{a}$ and $\bar{b}$ be the vectors along the lines whose direction ratios are 1, 3, 2 and - 1, 1, 2 respectively.

Then $\bar{a} = \hat{i} + 3\hat{j} + 2\hat{k}$ and $\bar{b} = -\hat{i} + \hat{j} + 2\hat{k}$

The vector perpendicular to both $\bar{a}$ and $\bar{b}$ is given by

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{vmatrix}$$

$= (6 - 2)\hat{i} - (2 + 2)\hat{j} - (-3 - 1)\hat{k}$

$= 4\hat{i} - 4\hat{j} + 4\hat{k}$

Hence, the required direction ratios are 4, -4, 4.

Exercise 5.4 | Q 17 | Page 179

Prove that the two vectors whose direction cosines are given by relations al + bm + cn = 0 and fmn + gnl + hlm = 0 are perpendicular, if $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$

Solution:

Given: al + bm + cn = 0 ...(1)

and fmn + gnl + hlm = 0 ...(2)

From (1), n = - $\left(\frac{al + bm}{c}\right)$ ...(3)

Substituting this value of n in equation (2), we get

(fm + gl). $\left[-\frac{al + bm}{c}\right] + hlm = 0$

∴ - (aflm + bfm^2 + agl^2 + bglm) + chlm = 0

∴ agl^2 + (af + bg - ch)lm + bfm^2 = 0 ...(4)

Note that both l and m cannot be zero, because if l = m = 0, then from (3), we get n = 0, which is not possible as $l^2 + m^2 + n^2 = 1$

Let us take m ≠ 0.

Dividing equation (4) by m^2, we get

$$ag\left(\frac{1}{m^2}\right)+(af+bg-ch)\left(\frac{1}{m}\right)+bf=0 \quad \text{....(5)}$$

This is quadratic equation in $\left(\frac{1}{m}\right)$

If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of the two lines given by the equation (1) and (2), then

$\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$ are the roots of the equation (5).

From the quadratic equation (5), we get

product of roots = $\frac{l_1}{m_1}\cdot\frac{l_2}{m_2}=\frac{bf}{ag}$

$$\therefore \frac{l_1 l_2}{m_1 m_2}=\frac{f/a}{g/b}$$

$$\therefore \frac{l_1 l_2}{f/a}=\frac{m_1 m_2}{g/b}$$

Similarly, we can show that,

$$\frac{l_1 l_2}{f/a}=\frac{n_1 n_2}{h/c}$$

$$\therefore \frac{l_1 l_2}{f/a}=\frac{m_1 m_2}{g/b}=\frac{n_1 n_2}{h/c}=\lambda \quad \text{...(Say)}$$

$$\therefore l_1l_2 = \lambda\left(\frac{f}{a}\right), m_1m_2 = \lambda\left(\frac{g}{b}\right), n_1n_2 = \lambda\left(\frac{h}{c}\right)$$

Now, the lines are perpendicular if

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

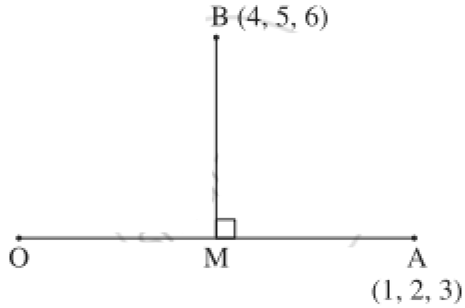
i.e. if $\lambda\left(\frac{f}{a}\right) + \lambda\left(\frac{g}{b}\right) + \lambda\left(\frac{h}{c}\right) = 0$

i.e. if $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$

Exercise 5.4 | Q 18 | Page 179

If A(1, 2, 3) and B(4, 5, 6) are two points, then find the foot of the perpendicular from the point B to the line joining the origin and the point A.

Solution:



Let M be the foot of the perpendicular drawn from B to the line joining O and A.

Let M = (x, y, z)

OM has direction ratios x - 0, y - 0, z - 0 = x, y, z

OA has direction ratios 1 - 0, 2 - 0, 3 - 0 = 1, 2, 3

But O, M, A are collinear.

$$\therefore \frac{x}{1} = \frac{y}{2} = \frac{z}{3} = k \quad \text{....(Let)}$$

∴ x = k, y = 2k, z = 3k

∴ m = (k, 2k, 3k)

BM has direction ratios

k - 4, 2k - 5, 3k - 6

∵ BM is perpendicular to OA.

∴ (1)(k - 4) + 2(2k - 5) + 3(3k - 6) = 0

∴ k - 4 + 4k - 10 + 9k - 18 = 0

∴ 14k = 32

$\therefore k = \dfrac{16}{7}$

$\therefore M = (k, 2k, 3k) = \left(\dfrac{16}{7}, \dfrac{32}{7}, \dfrac{48}{7}\right)$

[**Note:** Answer in the textbook is incorrect.]

EXERCISE 5.5 [PAGES 183 - 184]

Exercise 5.5 | Q 1 | Page 183

Find $\bar{a}.\left(\bar{b}\times\bar{c}\right)$ if $\bar{a} = 3\hat{i} - \hat{j} + 4\hat{k}, \bar{b} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\bar{c} = -5\hat{i} + 2\hat{j} + 3\hat{k}$

Solution:

$$\bar{a}.\left(\bar{b}\times\bar{c}\right) = \begin{vmatrix} 3 & -1 & 4 \\ 2 & 3 & -1 \\ -5 & 2 & 3 \end{vmatrix}$$

= 3(9 + 2) +1(6 - 5) + 4(4 + 15)

= 33 + 1 + 76

= 110

Exercise 5.5 | Q 2 | Page 183

If the vectors $3\hat{i} + 5\hat{k}, 4\hat{i} + 2\hat{j} - 3\hat{k}$ and $3\hat{i} + \hat{j} + 4\hat{k}$ are the coterminus edges of the parallelopiped, then find the volume of the parallelopiped.

Solution:

Let $\bar{a} = 3\hat{i} + 5\hat{k}, \bar{b} = 4\hat{i} + 2\hat{j} - 3\hat{k}, \bar{c} = 3\hat{i} + \hat{j} + 4\hat{k}$

$$\therefore \left[\bar{a}, \bar{b}, \bar{c}\right] = \begin{vmatrix} 3 & 0 & 5 \\ 4 & 2 & -3 \\ 3 & 1 & 4 \end{vmatrix}$$

= 3(8 + 3) - 0(16 + 9) + 5(4 - 6)

= 33 - 0 - 10

= 23

∴ volume of the parallelopiped = $\left[\bar{a}, \bar{b}, \bar{c}\right]$

= 23 cubic units.

Exercise 5.5 | Q 3 | Page 183

If the vectors $-3\hat{i} + 4\hat{i} - 2\hat{k}, \hat{i} + 2\hat{k}$ and $\hat{i} - p\hat{j}$ are coplanar, then find the value of p.

Solution:

Let $\bar{a} = -3\hat{i} + 4\hat{i} - 2\hat{k}, \bar{b} = \hat{i} + 2\hat{k}, \bar{c} = \hat{i} - p\hat{j}$

Then $\bar{a}, \bar{b}, \bar{c}$ are coplanar

$$\therefore \left[\bar{a}\ \bar{b}\ \bar{c}\right] = 0$$

$$\therefore \begin{vmatrix} -3 & 4 & -2 \\ 1 & 0 & 0 \\ 1 & -p & 0 \end{vmatrix} = 0$$

$\therefore$ -3(0 + 2p) - 4(0 - 2) - 2(- p - 0) = 0

$\therefore$ - 6p + 8 + 2p = 0

$\therefore$ - 4p = - 8

$\therefore$ p = 2.

Exercise 5.5 | Q 4.1 | Page 184

Prove that $\left[\bar{a}\ \bar{b}+\bar{c}\ \bar{a}+\bar{b}+\bar{c}\right]=0$

Solution:

$\left[\bar{a}\ \bar{b}+\bar{c}\ \bar{a}+\bar{b}+\bar{c}\right]$

$=\bar{a}.\left[\left(\bar{b}+\bar{c}\right)\times\left(\bar{a}+\bar{b}+\bar{c}\right)\right]$

$=\bar{a}.\left(\bar{b}\times\bar{a}+\bar{b}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a}+\bar{c}\times\bar{b}+\bar{c}\times\bar{c}\right)$

$=\bar{a}.\left(\bar{b}\times\bar{a}\right)+\bar{a}\left(\bar{b}\times\ \bar{b}\right)+\bar{a}.\left(\bar{b}\times\bar{c}\right)+\bar{a}.\left(\bar{c}\times\bar{a}\right)+\bar{a}.\left(\bar{c}\times\bar{b}\right)+\bar{a}.\left(\bar{c}\times\bar{c}\right)$

$=0+0+\bar{a}.\left(\bar{b}\times\bar{c}\right)+0-\bar{a}.\left(\bar{b}\times\bar{c}\right)+0$

= 0

Exercise 5.5 | Q 4.2 | Page 184

Prove that $\left(\bar{a}+2\bar{b}-\bar{c}\right).\left[\left(\bar{a}-\bar{b}\right)\times\left(\bar{a}-\bar{b}-\bar{c}\right)\right]=3\left[\bar{a}\bar{b}\bar{c}\right]$.

Solution:

$\left(\bar{a}+2\bar{b}-\bar{c}\right).\left[\left(\bar{a}-\bar{b}\right)\times\left(\bar{a}-\bar{b}-\bar{c}\right)\right]=3\left[\bar{a}\bar{b}\bar{c}\right]$

$=\left(\bar{a}+2\bar{b}-\bar{c}\right).\left(\bar{a}\times\bar{a}-\bar{a}\times\bar{b}-\bar{a}\times\bar{c}-\bar{b}\times\bar{a}+\bar{b}\times\bar{b}+\bar{b}\times\bar{c}\right)$

$=\left(\bar{a}+2\bar{b}-\bar{c}\right).\left(\bar{0}-\bar{a}\times\bar{b}-\bar{c}\times\bar{a}-\bar{a}\times\bar{b}+\bar{0}+\bar{b}\times\bar{c}\right)$

$=\left(\bar{a}+2\bar{b}-\bar{c}\right).\left(\bar{c}\times\bar{a}+\bar{b}\times\bar{c}\right)$

$=\bar{a}.\left(\bar{c}\times\bar{a}\right)+\bar{a}.\left(\bar{b}\times\bar{c}\right)+2\bar{b}.\left(\bar{c}\times\bar{a}\right)+2\bar{b}.\left(\bar{b}\times\bar{c}\right)-\bar{c}.\left(\bar{c}\times\bar{a}\right)-\bar{c}.\left(\bar{b}\times\bar{c}\right)$

$= 0 + \bar{a}.\left(\bar{b} \times \bar{c}\right) + 2\bar{b}.\left(\bar{c} \times \bar{a}\right) + 2 \times 0 - 0 - 0$

$= \left[\bar{a}\ \bar{b}\ \bar{c}\right] + 2\left[\bar{b}\ \bar{c}\ \bar{a}\right]$

$= \left[\bar{a}\ \bar{b}\ \bar{c}\right] + 2\left[\bar{a}\ \bar{b}\ \bar{c}\right] = 3\left[\bar{a}\ \bar{b}\ \bar{c}\right]$

Exercise 5.5 | Q 5 | Page 184

If $\bar{c} = 3\bar{a} - 2\bar{b}$, then prove that $\left[\bar{a}\bar{b}\bar{c}\right] = 0$

Solution:

We use the results: $\bar{b} \times \bar{b} = \bar{0}$ and if in a scalar triple product, two vectors are equal, then the scalar triple product is zero.

$\left[\bar{a}\bar{b}\bar{c}\right] = \bar{a}.\left(\bar{b} \times \bar{c}\right)$

$= \bar{a}.\left[\bar{b} \times \left(3\bar{a} - 2\bar{b}\right)\right]$

$= \bar{a}.\left(3\bar{b} \times \bar{a} - 2\bar{b} \times \bar{b}\right)$

$= \bar{a}.\left(3\bar{b} \times \bar{a} - \bar{0}\right)$

$= 3\bar{a}.\left(\bar{b} \times \bar{a}\right) = 3 \times 0 = 0$

Alternative Method:

$\bar{c} = 3\bar{a} - 2\bar{b}$

$\therefore$ $\bar{c}$ is a linear combination of $\bar{a}$ and $\bar{b}$

$\therefore$ $\bar{a}, \bar{b}, \bar{c}$ are coplanar

$\therefore$ $\left[\bar{a}, \bar{b}, \bar{c}\right] = 0.$

Exercise 5.5 | Q 6 | Page 184

If $\bar{u} = \hat{i} - 2\hat{j} + \hat{k}, \bar{r} = 3\hat{i} + \hat{k}$ and $\overline{w} = \hat{j} - \hat{k}$ are given vectors, then find $(\bar{u} + \overline{w}).[(\bar{u} \times \bar{r}) \times (\bar{r} \times \overline{w})]$

Solution:

$$\bar{u} + \overline{w} = \left(\hat{i} - 2\hat{j} + \hat{k}\right) + \left(\hat{j} - \hat{k}\right)$$

$$= \hat{i} - \hat{j}$$

$$\bar{u} \times \bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 1 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= (-2 - 0)\hat{i} - (1 - 3)\hat{j} + (0 + 6)\hat{k}$$

$$= -2\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\bar{r} \times \overline{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= (0 - 1)\hat{i} - (-3 - 0)\hat{j} + (3 - 0)\hat{k}$$

$$= -\hat{i} + 3\hat{j} + 3\hat{k}$$

Now, $(\bar{u} + \overline{w}).[(\bar{u} \times \bar{r}) \times (\bar{r} \times \overline{w})] = \begin{vmatrix} 1 & -1 & 0 \\ -2 & 2 & 6 \\ -1 & 3 & 3 \end{vmatrix}$

= 1(6 - 18) + 1(- 6 + 6) + 0

= - 12 + 0 + 0 = - 12

Exercise 5.5 | Q 7 | Page 184

Find the volume of a tetrahedron whose vertices are A (- 1, 2, 3), B (3, - 2, 1), C (2, 1, 3) and D (- 1, 2, 4).

Solution:

$\bar{a} = -\hat{i} + 2\hat{j} + 3\hat{k},\ \bar{b} = 3\hat{i} - 2\hat{j} + \hat{k}, \bar{c} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\bar{d} = -\hat{i} - 2\hat{j} + 4\hat{k}$

$\therefore \overline{AB} = \bar{b} - \bar{a}$

$= \left(3\hat{i} - 2\hat{j} + \hat{k}\right) - \left(-\hat{i} + 2\hat{j} + 3\hat{k}\right)$

$= 4\hat{i} - 4\hat{j} - 2\hat{k}$

$\overline{AC} = \bar{c} - \bar{a}$

$= \left(2\hat{i} + \hat{j} + 3\hat{k}\right) - \left(-\hat{i} + 2\hat{j} + 3\hat{k}\right)$

$= 3\hat{i} - \hat{j}$

$\overline{AD} = \bar{d} - \bar{a}$

$= \left(-\hat{i} - 2\hat{j} + 4\hat{k}\right) - \left(-\hat{i} + 2\hat{j} + 3\hat{k}\right)$

$= -4\hat{j} + \hat{k}$

$$\therefore \left[\overline{AB}\ \overline{AC}\ \overline{AD}\right] = \begin{vmatrix} 4 & -4 & -2 \\ 3 & -1 & 0 \\ 0 & -4 & 1 \end{vmatrix}$$

= 4(- 1 + 0) + 4(3 - 0) - 2(- 12 + 0)

= - 4 + 12 + 24 = 32

$\therefore$ volume of the tetrahedron = $\frac{1}{6}\left|\left[\overline{AB}\ \overline{AC}\ \overline{AD}\right]\right|$

$= \frac{1}{6} \times 32 = \frac{16}{3}$ cu units.

Exercise 5.5 | Q 8 | Page 184

If $\bar{a} = \hat{i} + 2\hat{j} + 3\hat{k}, \bar{b} = 3\hat{i} + 2\hat{j}$ and $\bar{c} = 2\hat{i} + \hat{j} + 3\hat{k}$, then verify that $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c}$

Solution:

$$\bar{b} \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 0 \\ 2 & 1 & 3 \end{vmatrix}$$

$$= (6 - 0)\hat{i} - (9 - 0)\hat{j} + (3 - 4)\hat{k}$$

$$= 6\hat{i} - 9\hat{j} - \hat{k}$$

$$\therefore \bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 6 & -9 & -1 \end{vmatrix}$$

$$= (-2 + 27)\hat{i} - (-1 - 18)\hat{j} + (-9 - 12)\hat{k}$$

$$= 25\hat{i} + 19\hat{j} - 21\hat{k} \quad ...(1)$$

$$\bar{a}.\bar{c} = (\hat{i} + 2\hat{j} + 3\hat{k}).(2\hat{i} + \hat{j} + 3\hat{k})$$

= (1)(2) + (2)(1) + (3)(3)

= 2 + 2 + 9 = 13

$$\therefore (\bar{a}.\bar{c}).\bar{b} = 13(3\hat{i} + 2\hat{j}) = 39\hat{i} + 26\hat{j}$$

Also, $(\bar{a}.\bar{b}) = (\hat{i} + 2\hat{j} + 3\hat{k}).(3\hat{i} + 2\hat{j})$

= (1)(3) + (2)(2) + (3)(0)

= 3 + 4 + 0 = 7

$$\therefore (\bar{a}.\bar{b}).\bar{c} = 7\left(2\hat{i} + \hat{j} + 3\hat{k}\right) = 14\hat{i} + 7\hat{j} + 21\hat{k}$$

$$\therefore (\bar{a}.\bar{c}).\bar{b} - (\bar{a}.\bar{b}).\bar{c}$$

$$= \left(39\hat{i} + 26\hat{j}\right) - \left(14\hat{i} + 7\hat{j} + 21\hat{k}\right)$$

$$= 25\hat{i} + 19\hat{j} - 21\hat{k} \quad .....(2)$$

From (1) and (2), we get

$$\bar{a} \times \left(\bar{b} \times \bar{c}\right) = (\bar{a}.\bar{c})\bar{b} - \left(\bar{a}.\bar{b}\right)\bar{c}$$

Exercise 5.5 | Q 9 | Page 184

If $\bar{a} = \hat{i} - 2\hat{j}$, $\bar{b} = \hat{i} + 2\hat{j}$, $\bar{c} = 2\hat{i} + \hat{j} - 2\hat{k}$, then find (i) $\bar{a} \times \left(\bar{b} \times \bar{c}\right)$ (ii) $\left(\bar{a} \times \bar{b}\right) \times \bar{c}$ Are the results same? Justify.

Solution:

$$\bar{a} \times \left(\bar{b} \times \bar{c}\right)$$

$$\bar{b} \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= (-4 - 0)\hat{i} - (-2 - 0)\hat{j} + (1 - 4)\hat{k}$$

$$= -4\hat{i} + 2\hat{j} - 3\hat{k}$$

$$\therefore \bar{a} \times \left(\bar{b} \times \bar{c}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 0 \\ -4 & 2 & -3 \end{vmatrix}$$

$$= (6 - 0)\hat{i} - (-3 - 0)\hat{j} + (2 - 8)\hat{k}$$

$$= 6\hat{i} + 3\hat{j} - 6\hat{k}$$

$(\bar{a} \times \bar{b}) \times \bar{c}$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 0 \\ 1 & 2 & 0 \end{vmatrix}$$

$= (0 - 0)\hat{i} - (0 - 0)\hat{j} + (2 - (-2))\hat{k}$

$= 4\hat{k}$

$$\therefore (\bar{a} \times \bar{b}) \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 4 \\ 2 & 1 & -2 \end{vmatrix}$$

$= (0 - 4)\hat{i} - (0 - 8)\hat{j} + (0 - 0)\hat{k}$

$= -4\hat{i} + 8\hat{j}$

$\bar{a} \times (\bar{b} \times \bar{c}) \neq (\bar{a} \times \bar{b}) \times \bar{c}$

Exercise 5.5 | Q 10 | Page 184

Show that $\bar{a} \times (\bar{b} \times \bar{c}) + \bar{b} \times (\bar{c} \times \bar{a}) + \bar{c} \times (\bar{a} \times \bar{b}) = \bar{0}$

Solution:

LHS = $\bar{a} \times (\bar{b} \times \bar{c}) + \bar{b} \times (\bar{c} \times \bar{a}) + \bar{c} \times (\bar{a} \times \bar{b})$

$= (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c} + (\bar{b}.\bar{a})\bar{c} - (\bar{b}.\bar{c})\bar{a} + (\bar{c}.\bar{b})\bar{a} - (\bar{c}.\bar{a})\bar{b}$

$= (\bar{c}.\bar{a})\bar{b} - (\bar{a}.\bar{b})\bar{c} + (\bar{a}.\bar{b})\bar{c} - (\bar{b}.\bar{c})\bar{a} + (\bar{b}.\bar{c})\bar{a} - (\bar{c}.\bar{a})\bar{b}$ .....

[∵ bar"a".bar"b" = bar"b".bar"a"]

= $\bar{0}$ = RHS

MISCELLANEOUS EXERCISE 5 [PAGES 187 - 189]

Miscellaneous exercise 5 | Q 1.01 | Page 187

Select the correct option from the given alternatives:

If $|\bar{a}| = 2, |\bar{b}| = 3, |\bar{c}| = 4$ then $[\bar{a}+\bar{b} \ \ \bar{b}+\bar{c} \ \ \bar{c}-\bar{a}]$ is equal to

1. 24
2. - 24
3. **0**
4. 48

Solution:

$[\bar{a}+\bar{b} \ \ \bar{b}+\bar{c} \ \ \bar{c}-\bar{a}]$ = **0**

Miscellaneous exercise 5 | Q 1.02 | Page 188

Select the correct option from the given alternatives:

If $|\bar{a}| = 3, |\bar{b}| = 4,$ then the value of λ for which $\bar{a} + \lambda\bar{b}$, is perpendicular to $\bar{a} - \lambda\bar{b}$, is

1. 9/16
2. **3/4**
3. 3/2
4. 4/3

Solution: 3/4

Miscellaneous exercise 5 | Q 1.03 | Page 188

Select the correct option from the given alternatives:

If sum of two unit vectors is itself a unit vector, then the magnitude of their difference is

1. √2
2. **√3**
3. 1
4. 2

Solution: √3

Miscellaneous exercise 5 | Q 1.04 | Page 188

Select the correct option from the given alternatives:

If $|\bar{a}| = 3, |\bar{b}| = 5, |\bar{c}| = 7$ and $\bar{a} + \bar{b} + \bar{c} = \bar{0}$, then the angle between $\bar{a}$ and $\bar{b}$ is

1. π/2
2. **π/3**
3. π/4
4. π/6

Solution: π/3

Miscellaneous exercise 5 | Q 1.05 | Page 188

Select the correct option from the given alternatives:

The volume of tetrahedron whose vectices are (1,-6,10), (-1, -3, 7), (5, -1, λ) and (7, -4, 7) is 11 cu units, then the value of λ is

1. **7**
2. 2
3. 1
4. 5

Solution: 7

Miscellaneous exercise 5 | Q 1.06 | Page 188

Select the correct option from the given alternatives:

If α, β, γ are direction angles of a line and α = 60°, β = 45°, γ = ______.

1. 30° or 90°
2. 45° or 60°
3. 90° or 30°
4. **60° or 120°**

Solution: 60° or 120°

Miscellaneous exercise 5 | Q 1.07 | Page 188

Select the correct option from the given alternatives:

The distance of the point (3, 4, 5) from Y-axis is

1. 3

2. 5
3. **$\sqrt{34}$**
4. $\sqrt{41}$

Solution: $\sqrt{34}$

Miscellaneous exercise 5 | Q 1.08 | Page 188

Select the correct option from the given alternatives:

The line joining the points (2, 1, 8) and (a, b, c) is parallel to the line whose direction ratios are 6, 2, 3. The value of a, b, c are

1. **4, 3, - 5**
2. 1, 2, -13/2
3. 10, 5, -2
4. 3, 5, 11

Solution: 4, 3, - 5

Miscellaneous exercise 5 | Q 1.09 | Page 188

Select the correct option from the given alternatives:

If cos α, cos β, cos γ are the direction cosines of a line, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is

1. 1
2. **2**
3. 3
4. 4

Solution: 2

Miscellaneous exercise 5 | Q 1.1 | Page 188

Select the correct option from the given alternatives:

If l, m, n are direction cosines of a line then $\mathrm{l}\hat{\mathrm{i}} + \mathrm{m}\hat{\mathrm{j}} + \mathrm{n}\hat{\mathrm{k}}$ is

1. null vector
2. **the unit vector along the line**
3. any vector along the line
4. a vector perpendicular to the line

Solution: the unit vector along the line

Miscellaneous exercise 5 | Q 1.11 | Page 188

Select the correct option from the given alternatives:

If $|\bar{a}| = 3$ and - 1 ≤ k ≤ 2, then $|k\bar{a}|$ lies in the interval

1. **[0, 6]**
2. [-3, 6]
3. [3, 6]
4. [1, 2]

Solution: [0, 6]

Miscellaneous exercise 5 | Q 1.12 | Page 188

Select the correct option from the given alternatives:

Let α, β, γ be distinct real numbers. The points with position vectors $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k},\ \beta\hat{i} + \gamma\hat{j} + \alpha\hat{k},\ \ \gamma\hat{i} + \alpha\hat{j} + \beta\hat{k}$

1. are collinear
2. **form an equilateral triangle**
3. form a scalene triangle
4. form a right angled triangle

Solution: form an equilateral triangle

Miscellaneous exercise 5 | Q 1.13 | Page 189

Select the correct option from the given alternatives:

Let $\bar{p}$ and $\bar{q}$ be the position vectors of P and Q respectively, with respect to O and $|\bar{p}| = p, |\bar{q}| = q$. The points R and S divide PQ internally and externally in the ratio 2 : 3 respectively. If OR and OS are perpendicular; then

1. **$9p^2 = 4q^2$**
2. $4p^2 = 9q^2$
3. $9p = 4q$

4. 4p = 9q

Solution: $9p^2 = 4q^2$

Miscellaneous exercise 5 | Q 1.14 | Page 189

Select the correct option from the given alternatives:

The 2 vectors $\hat{j} + \hat{k}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ represents the two sides AB and AC respectively of a Δ ABC. The length of the median through A is

1. **√34/2**
2. √48/2
3. √18
4. of the median through A is

Solution: √34/2

Miscellaneous exercise 5 | Q 1.15 | Page 189

Select the correct option from the given alternatives:

If $\bar{a}$ and $\bar{b}$ are unit vectors, then what is the angle between $\bar{a}$ and $\bar{b}$ for $\sqrt{3}\bar{a} - \bar{b}$ to be a unit vector?

1. **30°**
2. 45°
3. 60°
4. 90°

Solution: 30°

Miscellaneous exercise 5 | Q 1.16 | Page 189

Select the correct option from the given alternatives:

If θ be the angle between any two vectors $\bar{a}$ and $\bar{b}$ then $|\bar{a}.\bar{b}| = |\bar{a} \times \bar{b}|$, when θ is equal to

1. 0

2. **π/4**
3. π/2
4. π

Solution: π/4

Miscellaneous exercise 5 | Q 1.17 | Page 189

Select the correct option from the given alternatives:

The value of $\hat{i}.\left(\hat{j}\times\hat{k}\right)+\hat{j}.\left(\hat{i}\times\hat{k}\right)+\hat{k}.\left(\hat{i}\times\hat{j}\right)$ is

1. 0
2. - 1
3. **1**
4. 3

Solution: 1

Miscellaneous exercise 5 | Q 1.18 | Page 189

Select the correct option from the given alternatives:

Let a, b, c be distinct non-negative numbers. If the vectors $a\hat{i}+a\hat{j}+c\hat{k}, \hat{i}+\hat{k}$ and $c\hat{i}+c\hat{j}+b\hat{k}$ lie in a plane, then c is

1. the arithmetic mean of a and b
2. **the geometric mean of a and b**
3. the harmonic man of a and b
4. 0

Solution: the geometric mean of a and b

Miscellaneous exercise 5 | Q 1.19 | Page 189

Select the correct option from the given alternatives:

Let $\bar{a}=\hat{i}-\hat{j},\bar{b}=\hat{j}-\hat{k},\bar{c}=\hat{k}-\hat{i}$. If $\bar{d}$ is a unit vector such that $\bar{a}.\bar{d}=0=\left[\bar{b}\bar{c}\bar{d}\right]$, then $\bar{d}$ equals

Options

$$\pm\frac{\hat{i}+\hat{j}-2\hat{k}}{\sqrt{6}}$$

$$\pm\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

$$\pm\frac{\hat{i}+\hat{j}-\hat{k}}{\sqrt{3}}$$

$$\pm\hat{k}$$

Solution:

$$\pm\frac{\hat{i}+\hat{j}-2\hat{k}}{\sqrt{6}}$$

Miscellaneous exercise 5 | Q 1.2 | Page 189

Select the correct option from the given alternatives:

If $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar unit vectors such that $\bar{a} \times (\bar{b} \times \bar{c}) = \frac{\bar{b}+\bar{c}}{\sqrt{2}}$, then the angle between $\bar{a}$ and $\bar{b}$ is

1. **3π/4**
2. π/4
3. π/2
4. π

Solution: 3π/4

MISCELLANEOUS EXERCISE 5 [PAGES 190 - 193]

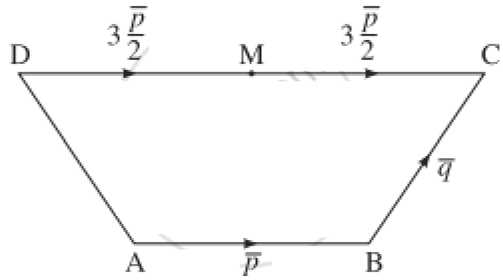
Miscellaneous exercise 5 | Q 1 | Page 190

ABCD is a trapezium with AB parallel to DC and DC = 3AB. M is the midpoint of DC. $\overline{AB} = \bar{p}, \overline{BC} = \bar{q}$.

Find in terms of $\bar{p}$ and $\bar{q}$:

(i) $\overline{AM}$ (ii) $\overline{BD}$ (iii) $\overline{MB}$ (iv) $\overline{DA}$

Solution:



DC is parallel to AB and DC = 3AB.

$\because \overline{AB} = \bar{p} \therefore \overline{DC} = 3\bar{p}$

M is the midpoint of DC

$\therefore \overline{DM} = \overline{MC} = \frac{1}{2}\overline{DC} = \frac{3}{2}\bar{p}$

(i) $\overline{AM} = \overline{AM} + \overline{BC} + \overline{CM}$

$= \overline{AB} + \overline{BC} - \overline{MC}$

$= \bar{p} + \bar{q} - \frac{3p}{2}$

$= \bar{q} - \frac{1}{2}\bar{p}$

(ii) $\overline{BD} = \overline{BC} + \overline{CD} = \overline{BC} - \overline{DC} = \bar{q} - 3\bar{p}$

(iii) $\overline{MB} = \overline{MC} + \overline{CB} = \overline{MC} - \overline{BC} = \frac{3}{2}\bar{p} - \bar{q}$

(iv) $\overline{DA} = \overline{DC} + \overline{CB} + \overline{BA} = \overline{DC} - \overline{BC} - \overline{AB}$

$$= 3\bar{p} - \bar{q} - \bar{p} = 2\bar{p} - \bar{q}$$

[**Note:** In the textbook answer instead of p and q, a and b are written.]

Miscellaneous exercise 5 | Q 2 | Page 190

The points A, B, C have position vectors $\bar{a}, \bar{b}$ and $\bar{c}$ respectively. The point P is the midpoint of AB. Find the vector $\overline{PC}$ in terms of $\bar{a}, \bar{b}, \bar{c}$.

Solution:

P is the mid-point of AB.

$\therefore \bar{p} = \dfrac{\bar{a} + \bar{b}}{2}$, where $\bar{p}$ is the position vector of P.

Now, $\overline{PC} = \bar{c} - \bar{p} = \bar{c} - \dfrac{1}{2}(\bar{a} + \bar{b})$

$= -\dfrac{1}{2}(\bar{a} + \bar{b}) + \bar{c}$

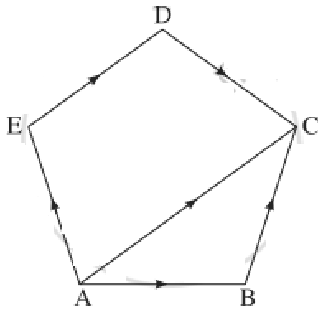
$= -\dfrac{1}{2}\bar{a} - \dfrac{1}{2}\bar{b} + \bar{c}$

Miscellaneous exercise 5 | Q 3 | Page 190

In a pentagon ABCDE, show that

$\overline{AB} + \overline{AE} + \overline{BC} + \overline{DC} + \overline{ED} = 2\overline{AC}$

Solution:



LHS = $\overline{AB} + \overline{AE} + \overline{BC} + \overline{DC} + \overline{ED}$

$= \left(\overline{AB} + \overline{BC}\right) + \left(\overline{AE} + \overline{ED} + \overline{DC}\right)$

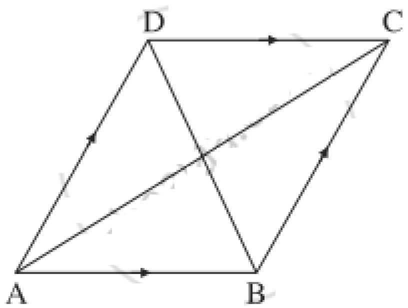
$= \overline{AC} + \overline{AC}$

$= 2\overline{AC}$ = RHS

Miscellaneous exercise 5 | Q 4 | Page 190

In a parallelogram ABCD, diagonal vectors are $\overline{AC} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\overline{BD} = -6\hat{i} + 7\hat{j} - 2\hat{k}$, then find the adjacent side vectors $\overline{AB}$ and $\overline{AD}$.

Solution:



ABCD is a parallelogram.

$\therefore \overline{AB} = \overline{DC}, \overline{AD} = \overline{BC}$

$\overline{AC} = \overline{AB} + \overline{BC}$

$= \overline{AB} + \overline{AD}$ ...(1)

$\overline{BD} = \overline{BA} + \overline{AD} = -\overline{AB} + \overline{AD}$ ...(2)

Adding (1) and (2), we get

$$2\overline{AD} = \overline{AC} + \overline{BD} = \left(2\hat{i} + 3\hat{j} + 4\hat{k}\right) + \left(-6\hat{i} + 7\hat{j} - 2\hat{k}\right)$$

$$= -4\hat{i} + 10\hat{j} + 2\hat{k}$$

$$\therefore \overline{AD} = \frac{1}{2}\left(-4\hat{i} + 10\hat{j} + 2\hat{k}\right)$$

$$= -2\hat{i} + 5\hat{j} + \hat{k}$$

From (1), $\overline{AB} = \overline{AC} - \overline{AD}$

$$= \left(2\hat{i} + 3\hat{j} + 4\hat{k}\right) - \left(-2\hat{i} + 5\hat{j} + \hat{k}\right)$$

$$= 4\hat{i} - 2\hat{j} + 3\hat{k}$$

Miscellaneous exercise 5 | Q 5 | Page 190

If two sides of a triangle are $\hat{i} + 2\hat{j}$ and $\hat{i} + \hat{k}$, find the length of the third side.

Solution:

Let ABC be a triangle with $\overline{AB} = \hat{i} + 2\hat{j}, \overline{BC} = \hat{i} + \hat{k}$.

By triangle law of vectors

$$\overline{AC} = \overline{AB} + \overline{BC}$$

$$= \left(\hat{i} + 2\hat{j}\right) + \left(\hat{i} + \hat{k}\right)$$

$$= 2\hat{i} + 2\hat{j} + \hat{k}$$

$\therefore$ l(AC) = $\left|\overline{AC}\right| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9}$ = 3 units

Hence, the length of third side is 3 units.

Miscellaneous exercise 5 | Q 6 | Page 190

If $|\bar{a}| = |\bar{b}| = 1,\ \bar{a}.\bar{b} = 0, \bar{a} + \bar{b} + \bar{c} = \bar{0}, \text{find } |\bar{c}|$.

Solution:

$\bar{a} + \bar{b} + \bar{c} = \bar{0}$

$\therefore -\bar{c} = \bar{a} + \bar{b}$

Taking dot product of both sides with itself, we get

$(-\bar{c}).(-\bar{c}) = (\bar{a} + \bar{b}).(\bar{a} + \bar{b})$

$\therefore |\bar{c}|^2 = \bar{a}.(\bar{a} + \bar{b}) + \bar{b}.(\bar{a} + \bar{b})$

$= \bar{a}.\bar{a} + \bar{a}.\bar{b} + \bar{b}\bar{a} + \bar{b}\bar{b}$

$= |\bar{a}|^2 + 0 + 0 + |\bar{b}|^2$ .... $[\bar{a}.\bar{b} = \bar{b}.\bar{a} = 0]$

= 1 + 1 = 2 ..... $[|\bar{a}| = |\bar{b}| = 1]$

$\therefore |\bar{c}| = \sqrt{2}$.

Miscellaneous exercise 5 | Q 7.1 | Page 190

Find the lengths of the sides of the triangle and also determine the type of a triangle:

A(2, -1, 0), B(4, 1, 1), C(4, -5, 4)

Solution:

The position vectors bar"a", bar"b", bar"c" of the points A, B, C are

$\bar{a} = 2\hat{i} - \hat{j}, \bar{b} = 4\hat{i} + \hat{j} + \hat{k}, \bar{c} = 4\hat{i} - 5\hat{j} + 4\hat{k}$

$\overline{AB} = \bar{b} - \bar{a} = \left(4\hat{i} + \hat{j} + \hat{k}\right) - \left(2\hat{i} - \hat{j}\right) = 2\hat{i} + 2\hat{j} + \hat{k}$

$\overline{BC} = \bar{c} - \bar{b} = \left(4\hat{i} - 5\hat{j} + 4\hat{k}\right) - \left(4\hat{i} + \hat{j} + \hat{k}\right) = -6\hat{j} + 3\hat{k}$

$\overline{CA} = \bar{a} - \bar{c} = \left(2\hat{i} - \hat{j}\right) - \left(4\hat{i} - 5\hat{j} + 4\hat{k}\right) = -2\hat{i} + 4\hat{j} - 4\hat{k}$

$\therefore$ l(AB) = $\left|\overline{AB}\right| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{4 + 4 + 1} = \sqrt{3} = 3$

l(BC) = $\left|\overline{BC}\right| = \sqrt{(-6)^2 + 3^2} = \sqrt{36 + 9} = \sqrt{45} = 3\sqrt{5}$

l(CA) = $\left|\overline{CA}\right| = \sqrt{(-2)^2 + 4^2 + (-4)^2} = \sqrt{4 + 16 + 16} = \sqrt{36} = 6$

$\therefore [l(AB)]^2 + [l(CA)]^2 = 3^2 + 6^2 = 9 + 36 = 45 = \left(3\sqrt{5}\right)^2$

$= [l(BC)]^2$

Miscellaneous exercise 5 | Q 7.2 | Page 190

Find the lengths of the sides of the triangle and also determine the type of a triangle:

L (3, -2, -3), M (7, 0, 1), N(1, 2, 1).

Solution:

The position vectors bar"a", bar"b", bar"c" of the points L, M, N are

$\bar{a} = 3\hat{i} - 2\hat{j} - 3\hat{k},\ \bar{b} = 7\hat{i} + \hat{k},\ \bar{c} = \hat{i} + 2\hat{j} + \hat{k}$

$\overline{LM} = \bar{b} - \bar{a} = \left(7\hat{i} + \hat{k}\right) - \left(3\hat{i} - 2\hat{j} - 3\hat{k}\right) = 4\hat{i} + 2\hat{j} + 4\hat{k}$

$\overline{MN} = \bar{c} - \bar{b} = \left(\hat{i} + 2\hat{j} + \hat{k}\right) - \left(7\hat{i} + \hat{k}\right) = -6\hat{i} + 2\hat{j}$

$\overline{NL} = \bar{a} - \bar{c} = \left(3\hat{i} - 2\hat{j} - 3\hat{k}\right) - \left(\hat{i} + 2\hat{j} + \hat{k}\right) = 2\hat{i} - 4\hat{j} - 4\hat{k}$

$\therefore$ l(LM) = $\left|\overline{AB}\right| = \sqrt{4^2 + 2^2 + 4^2} = \sqrt{16 + 4 + 16} = \sqrt{36} = 6$

l(MN) = $\left|\overline{MN}\right| = \sqrt{(-6)^2 + 2^2} = \sqrt{36 + 4} = \sqrt{40} = \sqrt{10 \times 4} = 2\sqrt{10}$

l(NL) = $\left|\overline{NL}\right| = \sqrt{(2)^2 + (-4)^2 + (-4)^2} = \sqrt{4 + 16 + 16} = \sqrt{36} = 6$

l(LM) = 6, l(MN) = $2\sqrt{10}$, l(NL) = 6

$\therefore \Delta$ LMN is isosceles.

Miscellaneous exercise 5 | Q 8.1 | Page 190

Find the component form of $\bar{a}$ if it lies in YZ-plane makes 60° with positive Y-axis and $|\bar{a}| = 4$.

Solution:

Let α, β, γ be the direction angles of $\bar{a}$

Since $\bar{a}$ lies in YZ-plane, , it is perpendicular to X-axis

∴ α = 90°

It is given that β = 60°

$\because \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore \cos^2 90° + \cos^2 60° + \cos^2\gamma = 1$

$\therefore 0 + \left(\frac{1}{2}\right)^2 + \cos^2\gamma = 1$

$\therefore \cos^2\gamma = 1 - \frac{1}{4} = \frac{3}{4}$

$\therefore \cos^2\gamma = \pm\frac{\sqrt{3}}{2}$

Unit vector along $\bar{a}$ is given by

$$\hat{a} = (\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k}$$

$$= 0.\,\hat{i} + \frac{1}{2}\hat{j} + \frac{\sqrt{3}}{2}\hat{k}$$

$$= \frac{1}{2}\hat{j} \pm \frac{\sqrt{3}}{2}\hat{k}$$

$\therefore \bar{a} = |\bar{a}|\hat{a} = 4\left(\frac{1}{2}\hat{j} \pm \frac{\sqrt{3}}{2}\hat{k}\right)$ .....[$\because |\bar{a}| = 4$]

$\therefore \bar{a} = 2\hat{j} \pm 2\sqrt{3}\hat{k}$

Miscellaneous exercise 5 | Q 9 | Page 190

Two sides of a parallelogram are $3\hat{i} + 4\hat{j} - 5\hat{k}$ and $-2\hat{j} + 7\hat{k}$.

Find unit vectors parallel to the diagonals.

Solution:

Let ABCD be a parallelogram with

$\overline{AB} = 3\hat{i} + 4\hat{j} - 5\hat{k}$ and $\overline{BC} = -2\hat{i} + 7\hat{k}$

Then $\overline{AC} = \overline{AB} + \overline{BC}$

$= \left(3\hat{i} + 4\hat{j} - 5\hat{k}\right) + \left(-2\hat{i} + 7\hat{k}\right)$

$= 3\hat{i} + 2\hat{j} + 2\hat{k}$

$\therefore \left|\overline{AC}\right| = \sqrt{3^2 + 2^2 + 2^2} = \sqrt{9 + 4 + 4} = \sqrt{17}$

$\therefore$ unit vector along $\overline{AC} = \frac{\overline{AC}}{\left|\overline{AC}\right|}$

$= \frac{1}{\sqrt{17}}\left(3\hat{i} + 2\hat{j} + 2\hat{k}\right)$

Also, $\overline{BD} = \overline{BA} + \overline{AD} = -\overline{AB} + \overline{BC} = \overline{BC} - \overline{AB}$

$= \left(-2\hat{i} + 7\hat{k}\right) - \left(3\hat{i} + 4\hat{j} - 5\hat{k}\right)$

$= -3\hat{i} - 6\hat{j} + 12\hat{k}$

$$= 3\left(-\hat{i} - 2\hat{j} + 4\hat{k}\right)$$

$$\therefore \left|\overline{BD}\right| = 3\sqrt{(-1)^2 + (-2)^2 + 4^2} = 3\sqrt{1 + 4 + 16} = 3\sqrt{21}$$

$\therefore$ unit vector along $\overline{BD} = \dfrac{\overline{BD}}{\left|\overline{BD}\right|}$

$$= \frac{3\left(-\hat{i} - 2\hat{j} + 4\hat{k}\right)}{3\sqrt{21}}$$

$$= \frac{1}{\sqrt{21}}\left(-\hat{i} - 2\hat{j} + 4\hat{k}\right)$$

Hence, the unit vectors parallel to the diagonals are

$$\frac{1}{\sqrt{17}}\left(3\hat{i} + 2\hat{j} + 2\hat{k}\right) \text{ and } \frac{1}{\sqrt{21}}\left(-\hat{i} - 2\hat{j} + 4\hat{k}\right)$$

Miscellaneous exercise 5 | Q 10 | Page 190

If D, E, F are the midpoints of the sides BC, CA, AB of a triangle ABC, prove that $\overline{AD} + \overline{BE} + \overline{CF} = \bar{0}$.

Solution:

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}$ be the position vectors of the points A, B, C, D, E, F respectively.

Since D, E, F are the midpoints of BC, CA, AB respectively, by the midpoint formula

$$\bar{d} = \frac{\bar{b} + \bar{c}}{2}, \bar{e} = \frac{\bar{c} + \bar{a}}{2}, \bar{f} = \frac{\bar{a} + \bar{b}}{2}$$

$$\therefore \overline{AD} + \overline{BE} + \overline{CF} = \left(\bar{d} - \bar{a}\right) + \left(\bar{e} - \bar{b}\right) + \left(\bar{f} - \bar{c}\right)$$

$$= \left(\frac{\bar{b} + \bar{c}}{2} - \bar{a}\right) + \left(\frac{\bar{c} + \bar{a}}{2} - \bar{b}\right) + \left(\frac{\bar{a} + \bar{b}}{2} - \bar{c}\right)$$

$$= \frac{1}{2}\bar{b} + \frac{1}{2}\bar{c} - \bar{a} + \frac{1}{2}\bar{c} + \frac{1}{2}\bar{a} - \bar{b} + \frac{1}{2}\bar{a} + \frac{1}{2}\bar{b} - \bar{c}$$

$$= \left(\bar{a} + \bar{b} + \bar{c}\right) - \left(\bar{a} + \bar{b} + \bar{c}\right) = \bar{0}.$$

Miscellaneous exercise 5 | Q 11 | Page 190

Find the unit vectors that are parallel to the tangent line to the parabola $y = x^2$ at the point (2, 4).

Solution:

Differentiating $y = x^2$ w.r.t. x, we get $\frac{dy}{dx} = 2x$

Slope of tangent at P(2, 4) = $\left(\frac{dy}{dx}\right)_{atP(2,4)}$ = 2 × 2 = 4

∴ the equation of tangent at P is

y - 4 = 4(x - 2)

∴ y = 4x - 4

∴ y = 4x is equation of line parallel to the tangent at P and passing through the origin O.

4x = y, z = 0

$$\therefore \frac{x}{1} = \frac{y}{4}, z = 0$$

∴ the direction ratios of this line are 1, 4, 0

∴ its direction cosines are

$$\pm\frac{1}{\sqrt{1^2 + 4^2 + 0^2}}, \pm\frac{4}{\sqrt{1^2 + 4^2 + 0^2}}, 0$$

i.e. $\pm\frac{1}{\sqrt{17}}, \pm\frac{4}{\sqrt{17}}, 0$

∴ unit vectors parallel to tangent line at P(2, 4) is

$$\pm\frac{1}{\sqrt{17}}\left(\hat{i}+4\hat{j}\right)$$

Miscellaneous exercise 5 | Q 12 | Page 190

Express $\hat{i}+4\hat{j}-4\hat{k}$ as the linear combination of the vectors $2\hat{i}-\hat{j}+3\hat{k}, \hat{i}-2\hat{j}+4\hat{k}$ and $-\hat{i}+3\hat{j}-5\hat{k}$.

Solution:

Let $\bar{a}=2\hat{i}-\hat{j}+3\hat{k}$,
$\bar{b}=\hat{i}-2\hat{j}+4\hat{k}$,
$\bar{c}=-\hat{i}+3\hat{j}-5\hat{k}$
$\bar{p}=\hat{i}+4\hat{j}-4\hat{k}$

Suppose $\bar{p}=x\bar{a}+y\bar{b}+z\bar{c}$.

Then, $\hat{i}+4\hat{j}-4\hat{k}=x\left(2\hat{i}-\hat{j}+3\hat{k}\right)+y\left(\hat{i}-2\hat{j}+4\hat{k}\right)+z\left(-\hat{i}+3\hat{j}-5\hat{k}\right)$

$\therefore\ \hat{i}+4\hat{j}-4\hat{k}=(2x+2y-z)\hat{i}+(-x-2y+3z)\hat{j}+(3x+4y-5z)\hat{k}$

By equality of vectors,

2x + 2y - z = 1

- x - 2y + 3z = 4

3x + 4y - 5z = - 4

We have to solve these equations by using Cramer's Rule.

D = $\begin{vmatrix} 2 & 2 & -1 \\ -1 & -2 & 3 \\ 3 & 4 & -5 \end{vmatrix}$

= 2(10 - 12) - 2(5 - 9) - 1 (-4 + 6)

= - 4 + 8 - 2

= 2 ≠ 0

$D_x = \begin{vmatrix} 1 & 2 & -1 \\ 4 & -2 & 3 \\ -4 & 4 & -5 \end{vmatrix}$

= 1(10 - 12) - 2(- 20 + 12) - 1(16 - 8)

= - 2 + 16 - 8

= 6

$D_y = \begin{vmatrix} 2 & 1 & -1 \\ -1 & 4 & 3 \\ 3 & -4 & -5 \end{vmatrix}$

= 2(- 20 + 12) - 1(5 - 9) - 1(4 - 12)

= - 16 - 4 - 8

= - 28

$D_z = \begin{vmatrix} 2 & 2 & 1 \\ -1 & -2 & 4 \\ 3 & 4 & -4 \end{vmatrix}$

= 2(8 - 16) - 2(4 - 12) + 1(- 4 + 6)

= - 16 - 16 + 2

= - 30

$\therefore x = \frac{D_x}{D} = \frac{6}{2} = 3$

$\therefore y = \frac{D_y}{D} = \frac{-28}{2} = -14$

$\therefore$ z = $\frac{D_z}{D} = \frac{-30}{2} = -15$

$\therefore \bar{p} = 3\bar{a} - 14\bar{b} - 3\bar{c}$

Miscellaneous exercise 5 | Q 13 | Page 190

If $\overline{OA} = \bar{a}$ and $\overline{OB} = \bar{b}$, then show that the vector along the angle bisector of ∠AOB is given by $\bar{d} = \lambda\left(\frac{\bar{a}}{|\bar{a}|} + \frac{\bar{b}}{|\bar{b}|}\right)$.

Solution: Choose any point P on the angle bisector of ∠AOB. Draw PM parallel to OB.

∴ ∠OPM = ∠POM = ∠POB

Hence, OM = MP

∴ OM and MP is the same scalar multiple of unit vectors $\hat{a}$ and $\hat{b}$ along these directions,

where $\hat{a} = \frac{\bar{a}}{|\bar{a}|}$ and $\hat{b} = \frac{\bar{b}}{|\bar{b}|}$

$\therefore \overline{OM} = \lambda\hat{a}$ and $\overline{MP} = \lambda\hat{b}$

$\therefore \overline{OP} = \overline{OM} + \overline{MP}$

$= \lambda\hat{a} + \lambda\hat{b}$

$= \lambda\left(\hat{a} + \hat{b}\right)$

Hence, the vector along angle bisector of ∠AOB is given by

$$\bar{d} = \overline{OP} = \lambda\left(\frac{\bar{a}}{|\bar{a}|} + \frac{\bar{b}}{|\bar{b}|}\right)$$

Miscellaneous exercise 5 | Q 15 | Page 191

A point P with position vector $\dfrac{-14\hat{i}+39\hat{j}+28\hat{k}}{5}$ divides the line joining A (1, 6, 5) and B in the ratio 3 : 2, then find the point B.

Solution: Let A, B and P have position vectors a, b and p respectively.

Then $\bar{a} = -\hat{i}+6\hat{j}+5\hat{k}$,

$$\bar{p} = \frac{-14\hat{i}+39\hat{j}+28\hat{k}}{5}$$

Now, P divides AB internally in the ratio 3 : 2

$$\therefore \bar{p} = \frac{3\bar{b}+2\bar{a}}{5}$$

$$\therefore 5\bar{p} = 3\bar{b}+2\bar{a}$$

$$\therefore 3\bar{b} = 5\bar{p}-2\bar{a}$$

$$\therefore 3\bar{b} = 5\left(\frac{-14\hat{i}+39\hat{j}+28\hat{k}}{5}\right)-2\left(-\hat{i}+6\hat{j}+5\hat{k}\right)$$

$$= -14\hat{i}+39\hat{j}+28\hat{k}+2\hat{i}-12\hat{j}-10\hat{k}$$

$$= -12\hat{i}+27\hat{j}+18\hat{k}$$

$$\therefore \bar{b} = -4\hat{i}+9\hat{j}+6\hat{k}$$

∴ coordinates of B are (-4, 9, 6).

Miscellaneous exercise 5 | Q 16 | Page 191

Show that the sum of three vectors determined by the medians of a triangle directed from the vertices is zero.

Solution:

Let $\vec{a}, \vec{b}$ and $\vec{c}$ are the position vectors of the vertices A, B and C respectively.

Then we know that the position vector of the centroid O of the triangle is $\dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$

Therefore sum of the three vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$, is

Therefore sum of the three vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$, is

$$\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\vec{a}-\left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right)+\vec{b}-\left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right)+\vec{c}-\left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right)$$

$$=\left(\vec{a}+\vec{b}+\vec{c}\right)-3\left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right)$$

$$=\vec{0}$$

Hence, Sum of the three vectors determined by the medians of a triangle directed from the vertices is zero.

Miscellaneous exercise 5 | Q 17 | Page 191

ABCD is a parallelogram. E, F are the midpoints of BC and CD respectively. AE, AF meet the diagonal BD at Q and P respectively. Show that P and Q trisect DB.

Solution:

Let A, B, C, D, E, F, P, Q have position vectors $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{p}, \bar{q}$ respectively.

$\because$ ABCD is a parallelogram

$\therefore \overline{AB}=\overline{DC}$

$\therefore \bar{b}-\bar{a}=\bar{c}-\bar{d}$

$\therefore \bar{c}=\bar{b}+\bar{d}-\bar{a}$ ....(1)

E is the midpoint of BC

$\therefore \bar{e}=\dfrac{\bar{b}+\bar{c}}{2}$

$\therefore 2\bar{e} = \bar{b} + \bar{c}$ ....(2)

F is the mid-point of CD

$$\therefore \bar{f} = \frac{\bar{c} + \bar{d}}{2}$$

$\therefore 2\bar{f} = \bar{c} + \bar{d}$ .....(3)

$2\bar{e} = \bar{b} + \bar{c}$ ...[By (2)]

$= \bar{b} + \left(\bar{b} + \bar{d} + \bar{a}\right)$ ....[By (1)]

$\therefore 2\bar{e} + \bar{a} = 2\bar{b} + \bar{d}$

$$\therefore \frac{2\bar{e} + \bar{a}}{2 + 1} = \frac{2\bar{b} + \bar{d}}{2 + 1}$$

LHS is the position vector of the point on AE and RHS is the position vector of the point on DB. But AE and DB meet at Q.

$$\therefore \bar{q} = \frac{2\bar{b} + \bar{d}}{2 + 1}$$

$\therefore$ Q divides DB in the ratio 2 : 1 ....(4)

$2\bar{f} = \bar{c} + \bar{d}$ ....[By(3)]

$= \left(\bar{b} + \bar{d} - \bar{a}\right) + \bar{d}$ ....[By(1)]

$\therefore 2\bar{f} + \bar{a} = 2\bar{d} + \bar{b}$

$$\therefore \frac{\bar{a} + 2\bar{f}}{1 + 2} = \frac{\bar{b} + 2\bar{d}}{1 + 2}$$

LHS is the position vector of the point on AF and RHS is the position vector of the point on DB. But AF and DB meet at P.

$$\therefore \bar{p} = \frac{\bar{b} + 2\bar{d}}{1 + 2}$$

∴ P divides DB in the ratio 1 : 2 .....(5)

From (4) and (5), if follows that P and Q trisect DB.

Miscellaneous exercise 5 | Q 18 | Page 191

If ABC is a triangle whose orthocentre is P and the circumcentre is Q, prove that $\overline{PA} + \overline{PB} + \overline{PC} = 2\overline{PQ}$.

Solution:

Let G be the centroid of the Δ ABC.

Let A, B, C, G, Q have position vectors $\bar{a}, \bar{b}, \bar{c}, \bar{g}, \bar{q}$ w.r.t. P. We know that Q, G, P are collinear and G divides segment QP internally in the ratio 1 : 2.

$$\therefore \bar{g} = \frac{1.\bar{p} + 2\bar{q}}{1 + 2} = \frac{2\bar{q}}{3} \quad \text{.........} [\because \bar{p} = \bar{0}]$$

$$\therefore 3\bar{g} = 2\bar{q}$$

$$\therefore \frac{3\left(\bar{a} + \bar{b} + \bar{c}\right)}{3} = 2\bar{q}$$

$$\therefore \bar{a} + \bar{b} + \bar{c} = 2\bar{q}$$

$$\therefore \overline{PA} + \overline{PB} + \overline{PC} = 2\overline{PQ}$$

Miscellaneous exercise 5 | Q 19 | Page 191

If P is orthocentre, Q is the circumcentre and G is the centroid of a triangle ABC, then prove that $\overline{QP} = 3\overline{QG}$.

Solution:

Let $\bar{p}$ and $\bar{g}$ be the position vectors of P and G w.r.t. the circumcentre Q.

i.e. $\overline{QR} = \bar{p}$ and $\overline{QG} = \bar{g}$

We know that Q, G, P are collinear and G divides segment QP internally in the ratio 1 : 2.

∴ by section formula for internal division,

$$\bar{g} = \frac{1.\bar{p} + 2\bar{q}}{1+2} = \frac{\bar{p}}{3} \qquad .....[\because \bar{q} = 0]$$

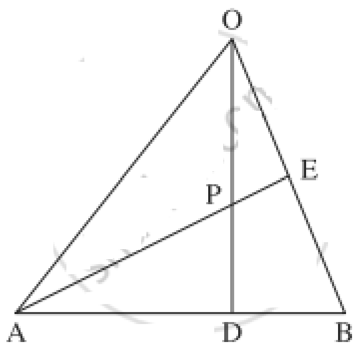
$$\therefore \bar{p} = 3\bar{g}$$

$$\therefore \overline{QP} = 3\overline{QG}.$$

Miscellaneous exercise 5 | Q 20 | Page 191

In Δ OAB, E is the midpoint of OB and D is the point on AB such that AD : DB = 2 : 1. If OD and AE intersect at P, then determine the ratio OP : PD using vector methods.

Solution:



Let A, B, D, E, P have position vectors $\bar{a}, \bar{b}, \bar{d}, \bar{e}, \bar{p}$ respectively w.r.t. O.

∵ AD : DB = 2 : 1

∴ D divides AB internally in the ratio 2 : 1.

Using section formula for internal division, we get

$\bar{d} = \frac{2\bar{b} + \bar{a}}{2+1}$

$\therefore 3\bar{d} = 2\bar{b} + \bar{a}$ ...(1)

Since E is the midpoint of OB, $\bar{e} = \overline{OE} = \frac{1}{2}\overline{OB} = \frac{\bar{b}}{2}$

$\therefore \bar{b} = 2\bar{e}$ ....(2)

∴ from (1),

$3\bar{d} = 2(2\bar{e}) + \bar{a}$ ...[By(2)]

$= 4\bar{e} + \bar{a}$

$\therefore \frac{3\bar{d} + 2.\bar{0}}{3+2} = \frac{4\bar{e} + \bar{a}}{4+1}$

LHS is the position vector of the point which divides OD internally in the ratio 3 : 2.

RHS is the position vector of the point which divides AE internally in the ratio 4 : 1.

But OD and AE intersect at P

∴ P divides OD internally in the ratio 3 : 2.

Hence, OP : PD = 3 : 2.

Miscellaneous exercise 5 | Q 21 | Page 191

Dot product of a vector with vectors

$3\hat{i} - 5\hat{k},\ 2\hat{i} + 7\hat{j}$ and $\hat{i} + \hat{j} + \hat{k}$ are respectively -1, 6 and 5.

Find the vector.

Solution:

Let $\bar{a} = 3\hat{i} - 5\hat{k}, \bar{b} = 2\hat{i} + 7\hat{j}, \bar{c} = \hat{i} + \hat{j} + \hat{k}$

Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the required vector.

Then, $\bar{r}.\bar{a} = -1, \bar{r}.\bar{a} = 6, \bar{r}.\bar{c} = 5$

$\therefore \left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(3\hat{i} - 5\hat{k}\right) = -1$

$\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(2\hat{i} + 7\hat{j}\right) = 6$ and

$\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(\hat{i} + \hat{j} + \hat{k}\right) = 5$

∴ 3x - 5y = - 1 ....(1)

∴ 2x + 7y = 6 ....(2)

∴ x + y + z = 5 ....(3)

From (3), z = 5 - x - y

Substituting this value of z in (1), we get

∴ 3x - 5(5 - x - y) = - 1

∴ 8x + 5y = 24 ....(4)

Multiplying (2) by 4 and subtracting from (4), we get

8x + 5y - 4(2x + 7y) = 24 - 6 × 4

∴ - 23y = 0

∴ y = 0

Substituting y = 0 in (2), we get

∴ 2x = 6

∴ x = 3

Substituting x = 3 in (1), we get

∴ 3(3) - 5z = - 1

∴ - 5z = - 10

∴ z = 2

$\therefore \bar{r} = 3\hat{i} + 0.\hat{j} + 2\hat{k}$

$= 3\hat{i} + 2\hat{k}$

Hence, the required vector is $3\hat{i} + 2\hat{k}$.

Miscellaneous exercise 5 | Q 22 | Page 191

If $\bar{a}, \bar{b}, \bar{c}$ are unit vectors such that $\bar{a} + \bar{b} + \bar{c} = \bar{0}$, then find the value of $\bar{a}.\bar{b} + \bar{b}.\bar{c} + \bar{c}.\bar{a}$.

Solution:

$\bar{a}, \bar{b}, \bar{c}$ are unit vectors

$\therefore |\bar{a}| = |\bar{b}| = |\bar{c}| = 1.$

Also, $\bar{a}.\bar{a} = \bar{b}.\bar{b} = \bar{c}.\bar{c} = 1$

Now, $\bar{a} + \bar{b} + \bar{c} = \bar{0}$ ...(1)

Taking scalar product of both sides with `bar"a", we get

$\bar{a}.(\bar{a} + \bar{b} + \bar{c}) = \bar{a}.\bar{0}$

$\therefore \bar{a}.\bar{a} + \bar{a}.\bar{b} + \bar{a}.\bar{c} = 0$

$\therefore \bar{a}.\bar{b} + \bar{a}.\bar{c} = -\bar{a}.\bar{a} = -1$ ....(2)

Similarly taking scalar product of both sides of (1) with $\bar{b}$ and $\bar{c}$, we get,

$\bar{b}.\bar{a} + \bar{b}.\bar{c} = -1$ ....(3)

$\bar{c}.\bar{a}+\bar{c}.\bar{b}=-1$ .....(4)

Adding (2), (3), (4) and using the fact that scalar product is commutative, we get

$2(\bar{a}.\bar{b}+\bar{b}.\bar{c}+\bar{c}.\bar{a})=-3$

$\therefore \bar{a}.\bar{b}+\bar{b}.\bar{c}+\bar{c}.\bar{a}=-\frac{3}{2}$

Miscellaneous exercise 5 | Q 23 | Page 191

If a parallelogram is constructed on the vectors $\bar{a}=3\bar{p}-\bar{q}, \bar{b}=\bar{p}+3\bar{q}$ and $|\bar{p}|=|\bar{q}|=2$ and angle between $\bar{p}$ and $\bar{q}$ is $\frac{\pi}{3}$, and angle between lengths of the sides is $\sqrt{7}:\sqrt{13}$.

Solution:

$|\bar{p}|=|\bar{q}|=2$ and angle between $\bar{p}$ and $\bar{q}$ is $\frac{\pi}{3}$.

$\therefore \bar{p}.\bar{q}=|\bar{p}||\bar{q}|\cos\frac{\pi}{3}=2\times 2\times\frac{1}{2}=2$

Now, $\bar{a}=3\bar{p}-\bar{q}$

$\therefore |\bar{a}|^2=|(3\bar{p}-\bar{q})|^2$

$=(3\bar{p}-\bar{q}).(3\bar{p}-\bar{q})$

$=3\bar{p}.(3\bar{p}-\bar{q})-\bar{q}.(3\bar{p}-\bar{q})$

$=9\bar{p}.\bar{p}-3\bar{p}.\bar{q}-3\bar{q}.\bar{p}+\bar{q}.\bar{q}$

$=9|\bar{p}|^2-6\bar{p}.\bar{q}+|\bar{q}|^2$ .....$[\because \bar{q}.\bar{p}=\bar{p}.\bar{q}]$

$=9\times 4-6\times 2+4$ .... $[\because \bar{p}\bar{q}=2]$

= 28

$\therefore |\bar{a}| = \sqrt{28}$

Also $\bar{b} = \bar{p} + 3\bar{q}$

$\therefore |\bar{b}|^2 = |\bar{p} + 3\bar{q}|^2$

$= (\bar{p} + 3\bar{q}).(\bar{p} + 3\bar{q})$

$= \bar{p}(\bar{p} + 3\bar{q}) + 3\bar{q}(\bar{p} + 3\bar{q})$

$= \bar{p}.\bar{p} + 3\bar{p}.\bar{q} - 3\bar{q}.\bar{p} + 9\bar{q}.\bar{q}$ ......$[\because \bar{p}.\bar{q} = \bar{q}.\bar{p}]$

$= |\bar{p}|^2 + 3\bar{p}q + 3\bar{p}.\bar{q} + 9|\bar{q}|^2$

= 4 + 12 + 36 ......$[\because \bar{p}.\bar{q} = 2]$

= 52

$\therefore |\bar{b}| = \sqrt{52}$

Ratio of lengths of the sides

$$= \frac{|\bar{a}|}{|\bar{b}|} = \frac{\sqrt{28}}{\sqrt{52}} = \frac{2\sqrt{7}}{2\sqrt{13}} = \frac{\sqrt{7}}{\sqrt{13}}.$$

Hence, the ratio of the lengths of the sides is $\sqrt{7} : \sqrt{13}$.

Miscellaneous exercise 5 | Q 24 | Page 191

Express the vector $\bar{a} = 5\hat{i} - 2\hat{j} + 5\hat{k}$ as a sum of two vectors such that one is parallel to the vector $\bar{b} = 3\hat{i} + \hat{k}$ and other is perpendicular to $\bar{b}$.

Solution:

Let $\bar{a} = \bar{c} + \bar{d}$, where $\bar{c}$ is parallel to $\bar{b}$ and $\bar{d}$ is perpendicular to $\bar{b}$.

Since, $\bar{c}$ is parallel to $\bar{b}$, $\bar{c} = m\bar{b}$, where m is a scalar.

$\therefore \bar{c} = m\left(3\hat{i} + \hat{k}\right)$

i.e. $\bar{c} = 3m\hat{i} + m\hat{k}$

Let $\bar{d} = x\hat{i} + y\hat{j} + z\hat{k}$

Since, $\bar{d}$ is perpendicular to $\bar{b} = 3\hat{i} + \hat{k}, \bar{d}.\bar{b} = 0$

$\therefore \left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(3\hat{i} + \hat{k}\right) = 0$

$\therefore 3x + z = 0$

$\therefore z = -3x$

$\therefore \bar{d} = x\hat{i} + y\hat{k} - 3x\hat{k}$

Now, $\bar{a} = \bar{c} + \bar{d}$ gives

$\therefore 5\hat{i} - 2\hat{j} + 5\hat{k} = \left(3m\hat{i} + m\hat{k}\right) + \left(x\hat{i} + y\hat{j} - 3x\hat{k}\right)$

$= (3m + x)\hat{i} + y\hat{j} + (m - 3x)\hat{k}$

By equality of vectors

3m + x = 5 ....(1)

y = - 2

and m - 3x = 5 ......(2)

From (1) and (2)

3m + x = m - 3x

∴ 2m = - 4x

∴ m = - 2x

Substituting m = - 2x in (1), we get

∴ - 6x + x = 5

∴ - 5x = 5

∴ x = - 1

∴ m = - 2x = 2

∴ $\bar{c} = 6\hat{i} + 2\hat{k}$ is parallel to $\bar{b}$ and $\bar{d} = -\hat{i} - 2\hat{j} + 3\hat{k}$ is perpendicular to $\bar{b}$

Hence, $\bar{a} = \bar{c} + \bar{d}$, where $\bar{c} = 6\hat{i} + 2\hat{k}$ and $\bar{d} = -\hat{i} - 2\hat{j} + 3\hat{k}$

Miscellaneous exercise 5 | Q 25 | Page 191

Find two unit vectors each of which makes equal angles with bar"u", bar"v" and bar"w" where bar"u" = 2hat"i" + hat"j" - 2hat"k", bar"v" = hat"i" + 2hat"j" - 2hat"k", bar"w" = 2hat"i" - 2hat"j" + hat"k".

Solution:

Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the unit vector which makes angle θ with each of the vectors

Then $|\bar{r}| = 1$

Also, $\bar{u} = 2\hat{i} + \hat{j} - 2\hat{k}, \bar{v} = \hat{i} + 2\hat{j} - 2\hat{k}, \overline{w} = 2\hat{i} - 2\hat{j} + \hat{k}$

$$|\bar{u}| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$$

$$|\bar{v}| = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$$

$$|\overline{w}| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

Angle between $\bar{r}$ and $\bar{u}$ is θ

$$\therefore \cos\theta = \frac{\bar{r}.\bar{u}}{|\bar{r}||\bar{u}|}$$

$$= \frac{\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(2\hat{i} + \hat{j} - 2\hat{k}\right)}{1 \times 3}$$

$$= \frac{2x + y - 2z}{3} \qquad \text{....(1)}$$

Also, the angle between $\bar{r}$ and $\bar{v}$ and between $\bar{r}$ and $\bar{w}$ is θ.

$$\therefore \cos\theta = \frac{\bar{r}.\bar{u}}{|\bar{r}||\bar{u}|}$$

$$= \frac{\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(\hat{i} + 2\hat{j} - 2\hat{k}\right)}{1 \times 3}$$

$$= \frac{x + 2y - 2z}{3} \qquad \text{....(2)}$$

and $\cos\theta = \frac{\bar{r}.\bar{u}}{|\bar{r}||\bar{u}|}$

$$= \frac{\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(2\hat{i} - 2\hat{j} + \hat{k}\right)}{1 \times 3}$$

$$= \frac{2x - 2y + z}{3} \qquad \text{....(3)}$$

From (1) and (2), we get

$$\frac{2x + y - 2z}{3} = \frac{x + 2y - 2z}{3}$$

∴ 2x + y - 2z = x + 2y - 2z

∴ x = y

From (2) and (3), we get

$$\frac{x + 2y - 2z}{3} = \frac{2x - 2y + z}{3}$$

∴ x + 2y - 2z = 2x - 2y + z

$\therefore$ 3y = 3z .....[$\because$ x = y]

$\therefore$ y = z

$\therefore$ x = y = z

$$\therefore \bar{r} = x\hat{i} + y\hat{j} + z\hat{k} = x\hat{i} + x\hat{j} + x\hat{k}$$

$$\therefore |\bar{r}| = \sqrt{x^2 + x^2 + x^2} = 1$$

$$\therefore x^2 + x^2 + x^2 = 1$$

$$\therefore 3x^2 = 1$$

$$\therefore x^2 = \frac{1}{3}$$

$$\therefore x = -\frac{1}{\sqrt{3}}$$

$$\therefore \bar{r} = \pm\frac{1}{\sqrt{3}}\hat{i} \pm \frac{1}{\sqrt{3}}\hat{j} \pm \frac{1}{\sqrt{3}}\hat{k}$$

$$= \pm\frac{1}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right)$$

Hence, the required unit vectors are $\pm\frac{1}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right)$

Miscellaneous exercise 5 | Q 26 | Page 191

Find the acute angle between the curves at their points of intersection, $y = x^2$, $y = x^3$.

Solution: The angle between the curves is the same as the angle between their tangents at the points of intersection.
We find the points of intersection of $y = x^2$ ....(1) and $y = x^3$ .....(2)

From (1) and (2)

$x^3 = x^2$

$\therefore x^3 - x^2 = 0$

$\therefore x^2(x - 1) = 0$

$\therefore x = 0$ or $x = 1$

When $x = 0$, $y = 0$.

When $x = 1$, $y = 1$.

$\therefore$ the points of intersection are

O = (0, 0) and P = (1, 1)

For $y = x^2$, $\frac{dy}{dx} = 2x$

For $y = x^3$, $\frac{dy}{dx} = 3x^2$

Angle at O = (0, 0)

Slope of tangent to $y = x^2$ at O

$$= \left(\frac{dy}{dx}\right)_{\text{at } O(0,0)} = 2 \times 0 = 0$$

$\therefore$ equation of tangent to $y = x^2$ at O is $y = 0$.

Slope of tangent to $y = x^3$ at O = $\left(\frac{dy}{dx}\right)_{\text{at } O(0,0)} = 3 \times 0 = 0$

$\therefore$ equation of tangent to $y = x^3$ at P is $y = 0$.

$\therefore$ the tangents to both curves at (0, 0) are $y = 0$

$\therefore$ angle between them is 0.

Angle at P = (1, 1)

Slope of tangent to $y = x^2$ at P

$$= \left(\frac{dy}{dx}\right)_{at\ O(1,1)} = 2 \times 1 = 2$$

∴ equation of tangent to $y = x^2$ at P is $y - 1 = 2(x - 1)$

∴ $y = 2x - 1$

Slope of tangent to $y = x^2$ at P = $\left(\frac{dy}{dx}\right)_{at\ O(1,1)} = 3 \times 1^2 = 3$

∴ equation of tangent to $y = x^3$ at P is $y - 1 = 3(x - 1)$

∴ $y = 3x - 2$

We have to find angle between $y = 2x - 1$ and $y = 3x - 2$

Lines through origin parallel to these tagents are

$y = 2x$ and $y = 3x$

$$\therefore \frac{x}{1} = \frac{y}{2} \text{ and } \frac{x}{1} = \frac{y}{3}$$

These lines lie in XY-plane.

∴ the direction ratios of these lines are 1, 2, 0 and 1, 3, 0.

The angle θ between them is given by

$$\cos\theta = \frac{(1)(1) + (2)(3) + (0)(0)}{\sqrt{1^2 + 2^2 + 0^2}\sqrt{1^2 + 3^2 + 0^2}}$$

$$= \frac{1 + 6 + 0}{\sqrt{5}\sqrt{10}}$$

$$= \frac{7}{\sqrt{50}} = \frac{7}{5\sqrt{2}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{7}{5\sqrt{2}}\right)$$

Hence, the required angles are 0 and $\cos^{-1}\left(\frac{7}{5\sqrt{2}}\right)$

Miscellaneous exercise 5 | Q 27.1 | Page 191

Find the direction cosines and direction angles of the vector

$2\hat{\text{i}} + \hat{\text{j}} + 2\hat{\text{k}}$

Solution:

Let $\bar{\text{a}} = 2\hat{\text{i}} + \hat{\text{j}} + 2\hat{\text{k}}$

$|\bar{\text{a}}| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$

∴ unit vector along bar"a"

$= \hat{\text{a}} = \frac{\bar{\text{a}}}{|\bar{\text{a}}|} = \frac{2\hat{\text{i}} + \hat{\text{j}} + 2\hat{\text{k}}}{3} = \frac{2}{3}\hat{\text{i}} + \frac{1}{3}\hat{\text{j}} + \frac{2}{3}\hat{\text{k}}$

∴ its direction cosines are $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$.

If α, β, γ are the direction angles, then cos α = $\frac{2}{3}$, cos β = $\frac{1}{3}$, cos γ = $\frac{2}{3}$

∴ α = $\cos^{-1}\left(\frac{2}{3}\right), \beta = \cos^{-1}\left(\frac{1}{3}\right), \gamma = \cos^{-1}\left(\frac{2}{3}\right)$

Hence, direction cosines are $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$ and direction angles are $\cos^{-1}\left(\frac{2}{3}\right), \beta = \cos^{-1}\left(\frac{1}{3}\right), \gamma = \cos^{-1}\left(\frac{2}{3}\right)$

Miscellaneous exercise 5 | Q 28 | Page 191

Let bar"b" = 4hat"i" + 3hat"j" and bar"c" be two vectors perpendicular to each other in the XY-plane. Find the vector in the same plane having projection 1 and 2 along bar"b" and bar"c" respectively.

Solution:

$\bar{b} = 4\hat{i} + 3\hat{j}$

$\therefore \left|\bar{b}\right| = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = 5$

Let $\bar{c} = m\hat{i} + n\hat{j}$ be perpendicular to $\bar{b}$

Then $\bar{b}.\bar{c} = 0$

$\therefore \left(4\hat{i} + 3\hat{j}\right).\left(m\hat{i} + n\hat{j}\right) = 0$

$\therefore$ 4m + 3n = 0

$\therefore$ n = $-\frac{4m}{3}$

$\therefore \bar{c} = m\hat{i} - \frac{4m}{3}\hat{j} = \frac{m}{3}\left(3\hat{i} - 4\hat{j}\right)$

$\therefore \bar{c} = p\left(3\hat{i} - 4\hat{j}\right) \quad ..... \left[p = \frac{m}{3}\right]$

$\therefore |\bar{c}| = p\sqrt{3^2 + (-4)^2} = p\sqrt{9 + 16} = 5p$

Let $\bar{d} = x\hat{i} + y\hat{j}$ be the vector having projections 1 and 2 along $\bar{b}$ and $\bar{c}$.

$\therefore \frac{\bar{b}.\bar{d}}{|\bar{b}|} = 1$

$\therefore \frac{\left(4\hat{i} + 3\hat{j}\right).\left(x\hat{i} + y\hat{j}\right)}{5} = 1$

$\therefore$ 4x + 3y = 5 .....(1)

Also, $\frac{\bar{c}.\bar{d}}{|\bar{c}|} = 2$

$\therefore \frac{\left(3p\hat{i} - 4p\hat{j}\right).\left(x\hat{i} + y\hat{j}\right)}{5p} = 2$

$\therefore$ 3px - 4py = 10p

$\therefore$ 3x - 4y = 10

From (1), 3y = 5 - 4x

$\therefore$ y = $\frac{5-4x}{3}$

Substituting for y in (2), we get

$3x - 4\left(\frac{5-4x}{3}\right) = 10$

$\therefore$ 9x - 20 + 16x = 30

$\therefore$ 25x = 50

$\therefore$ x = 2

y = $\frac{5-4x}{3} = \frac{5-4(2)}{3} = -1$

$\therefore \bar{d} = 2\hat{i} - \hat{j}$

Hence, the required vector is $2\hat{i} - \hat{j}$.

Miscellaneous exercise 5 | Q 29 | Page 192

Show that no line in space can make angles π/6 and π/4 with X-axis and Y-axis.

Solution: Let, if possible, a line in space make angles π/6 and π/4 with X-axis and Y-axis.

$\therefore \alpha = \frac{\pi}{6}, \beta = \frac{\pi}{4}$

Let the line make angle γ with Z-axis

$\because \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore \cos^2\left(\frac{\pi}{6}\right) + \cos^2\left(\frac{\pi}{4}\right) + \cos^2\gamma = 1$

$\therefore \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\gamma = 1$

$\therefore \cos^2\gamma = 1 - \frac{3}{4} - \frac{1}{2} = -\frac{1}{4}$

This is not possible, because cos γ is real.

∴ $\cos^2\gamma$ cannot be negative.

Hence, there is no line in space which makes angles π/6 and π/4 with X-axis and Y-axis.

Miscellaneous exercise 5 | Q 30 | Page 192

Find the angle between the lines whose direction cosines are given by the equations

6mn - 2nl + 5lm = 0, 3l + m + 5n = 0.

Solution: Given 6mn - 2nl + 5lm = 0 ....(1)

3l + m + 5n = 0. ...(2)

From (2), m = - 3l - 5n

Putting the value of m in equation (1), we get,

⇒ 6n(- 3l - 5n) - 2nl + 5l(- 3l - 5n) = 0

⇒ $- 18nl - 30n^2 - 2nl - 15l^2 - 25nl = 0$

⇒ $- 30n^2 - 45nl - 15l^2 = 0$

⇒ $2n^2 + 3nl + l^2 = 0$

⇒ $2n^2 + 2nl + nl + l^2 = 0$

⇒ (2n + l)(n + l) = 0

∴ 2n + l = 0 OR n + l = 0

∴ l = - 2n OR l = - n

∴ l = - 2n

From (2), 3l + m + 5n = 0

∴ - 6n + m + 5n = 0

∴ m = n

i.e. (- 2n, n, n) = (-2, 1, 1)

∴ l = - n

∴- 3n + m + 5n = 0

∴ m = - 2n

i.e. (-n, - 2n, n) = (1, 2, -1)

$(a_1, b_1, c_1) = (-2, 1, 1)$ and $(a_2, b_2, c_2) = (1, 2, -1)$

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}.\sqrt{a_2^2 + b^2{}_-2 + c_2^2}}\right|$$

$$= \left|\frac{(2)(1) + (-1)(2) + (-1)(-1)}{\sqrt{(2)^2 + 1^2 + 1^2}.\sqrt{1^2 + 2^2 + (1)^2}}\right|$$

$$= \left|\frac{2 - 2 + 1}{\sqrt{6}.\sqrt{6}}\right|$$

$$= \left|-\frac{1}{6}\right| = \frac{1}{6}$$

$$\theta = \cos^{-1}\left(\frac{1}{6}\right)$$

Miscellaneous exercise 5 | Q 31 | Page 192

If Q is the foot of the perpendicular from P (2, 4, 3) on the line joining the point A (1, 2, 4) and B(3, 4, 5), find coordinates of Q.

Solution: Let Q(x, y, z) be the co-ordinates then equation of line AB.

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$$\frac{x - 1}{3 - 1} = \frac{y - 2}{4 - 2} = \frac{z - 4}{5 - 4}$$

$$\frac{x - 1}{2} = \frac{y - 2}{2} = \frac{z - 4}{1} = k$$

∴ x = 2k + 1 , y = 2k + 2, z = 2k + 4

General point on the line AB is 2k + 1, 2k + 2, k + 4

Let co-ordinate of Q be x = 2k + 1, y = 2k + 2, z = k + 4

dr's of line PQ is 2k + 1 - 2, 2k + 2 - 4, k + 4 - 3 i.e. 2k - 1, 2k - 2, k + 1

Since line PQ is perpendicular to line AB so,

∴ 2(2k - 1) + 2(2k - 2) + 1(k + 1) = 0

∴ 4k - 2 + 4k - 4 + k + 1 = 0

∴ 9k - 5 = 0

$\therefore k = \frac{5}{9}$

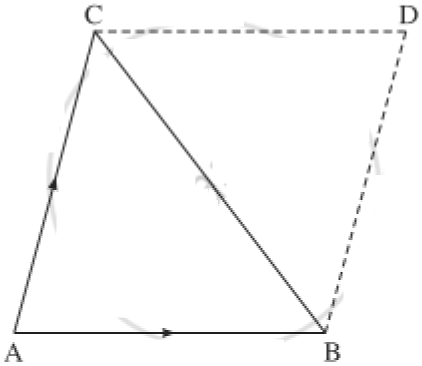
$\therefore x = 2 \times \frac{5}{9} + 1 = \frac{19}{9}$

$\left(\frac{19}{9}, \frac{28}{9}, \frac{41}{9}\right)$

Miscellaneous exercise 5 | Q 32 | Page 192

Show that the vector area of a triangle ABC, the position vectors of whose vertices are $\bar{a}, \bar{b}$ and $\bar{c}$ is $\frac{1}{2}\left[\bar{a} \times \bar{b} + \bar{b} \times \bar{c} + \bar{c} \times \bar{a}\right]$.

Solution:



Consider the triangle ABC.

Complete the parallelogram ABDC.

Vector area of Δ ABC.

$$= \frac{1}{2}(\text{vector area of parallelogram ABDC})$$

$$= \frac{1}{2}\left(\overline{AB} \times \overline{AC}\right)$$

$$= \frac{1}{2}\left[\left(\bar{b} - \bar{a}\right) \times \left(\bar{c} - \bar{a}\right)\right] \; ......\left[\because \overline{AB} = \bar{b} - \bar{a} \text{ and } \overline{AC} = \bar{c} - \bar{a}\right]$$

$$= \frac{1}{2}\left[\bar{b} \times \bar{c} - \bar{b} \times \bar{a} - \bar{a} \times \bar{c} + \bar{a} \times \bar{a}\right]$$

$$= \frac{1}{2}\left[\bar{b} \times \bar{c} + \bar{a} \times \bar{b} + \bar{c} \times \bar{a} + \bar{0}\right]$$

$$= \frac{1}{2}\left[\bar{a} \times \bar{b} + \bar{b} \times \ \bar{c} + \bar{c} \times \bar{a}\right]$$

Miscellaneous exercise 5 | Q 33 | Page 192

Find a unit vector perpendicular to the plane containing the point (a, 0, 0), (0, b, 0) and (0, 0, c). What is the area of the triangle with these vertices?

Solution:

The position vectors $\bar{p}, \bar{q}, \bar{r}$ of the points A(a, 0, 0), B(0, b, 0), C(0, 0, c) are

$$\bar{p} = a\hat{i}, \bar{q} = b\hat{j}, \bar{r} = c\hat{k}$$

$$\overline{AB} = \bar{q} - \bar{p} = b\hat{j} - a\hat{i} = -a\hat{j} + b\hat{j}$$

$$\overline{BC} = \bar{r} - \bar{q} = c\hat{k} - b\hat{j} = -b\hat{j} + c\hat{k}$$

$$\overline{AB} \times \overline{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a & b & 0 \\ 0 & -b & c \end{vmatrix}$$

$$= (bc - 0)\hat{i} - (-ac - 0)\hat{j} + (ab - 0)\hat{k}$$

$$= bc\hat{i} + ac\hat{j} + ab\hat{k}$$

$$\left|\overline{AB} \times \overline{BC}\right| = \sqrt{(bc)^2 + (ac)^2 + (ab)^2}$$

$$= \sqrt{b^2c^2 + a^2c^2 + a^2b^2}$$

$\overline{AB} \times \overline{BC}$ is perpendicular to the plane containing A, B, C.

$\therefore$ the required unit vector

$$= \frac{\overline{AB} \times \overline{BC}}{\left|\overline{AB} \times \overline{BC}\right|} = \frac{bc\hat{i} + ca\hat{j} + ab\hat{k}}{\sqrt{b^2c^2 + c^2a^2 + a^2b^2}}$$

Area of Δ ABC = $\frac{1}{2}\left|\overline{AB} \times \overline{BC}\right|$

$= \frac{1}{2}\sqrt{b^2c^2 + a^2c^2 + a^2b^2}$ sq.units.

Miscellaneous exercise 5 | Q 34.01 | Page 192

State whether the expression is meaningful. If not, explain why? If so, state whether it is a vector or a scalar:

$\bar{a}.\left(\bar{b} \times \bar{c}\right)$

Solution: This is the scalar product of two vectors. Therefore, this expression is meaningful and it is a **scalar.**

Miscellaneous exercise 5 | Q 34.02 | Page 192

State whether the expression is meaningful. If not, explain why? If so, state whether it is a vector or a scalar:

$\bar{a} \times \left(\bar{b}.\bar{c}\right)$

Solution:

This expression is meaningless because $\bar{a}$ is a vector, $\bar{b} \cdot \bar{c}$ is a scalar and vector product of vector and scalar is not defined.

Miscellaneous exercise 5 | Q 34.03 | Page 192

State whether the expression is meaningful. If not, explain why? If so, state whether it is a vector or a scalar:

$\bar{a} \times (\bar{b} \times \bar{c})$

Solution: This is vector product of two vectors. Therefore, this expression is meaningful and it is a **vector**.

Miscellaneous exercise 5 | Q 34.04 | Page 192

State whether the expression is meaningful. If not, explain why? If so, state whether it is a vector or a scalar:

$\bar{a}. (\bar{b}. \bar{c})$

Solution: This is meaningless because bar"a" is a vector, bar"b".bar"c" is a scalar and the scalar product of vector and scalar is not defined.

Miscellaneous exercise 5 | Q 34.05 | Page 192

State whether the expression is meaningful. If not, explain why? If so, state whether it is a vector or a scalar:

$(\bar{a}. \bar{b}) \times (\bar{c}. \bar{d})$

Solution:

This is meaningless because $\bar{a}. \bar{b}, \bar{c}. \bar{d}$ are scalars and cross product of two scalars is not defined.

Miscellaneous exercise 5 | Q 34.06 | Page 192

State whether the expression is meaningful. If not, explain why? If so, state whether it is a vector or a scalar:

$\left(\bar{a} \times \bar{b}\right) . \left(\bar{c} \times \bar{d}\right)$

Solution: This is scalar product of two vectors. Therefore, this expression is meaningful and it is a **scalar**.

Miscellaneous exercise 5 | Q 34.07 | Page 192

State whether the expression is meaningful. If not, explain why? If so, state whether it is a vector or a scalar:

$\left(\bar{a} . \bar{b}\right) . \bar{c}$

Solution:

This is meaningless because $\bar{c}$ is a vector, $\bar{a} . \bar{b}$ scalar and scalar product of vector and scalar is not defined.

Miscellaneous exercise 5 | Q 34.08 | Page 192

State whether the expression is meaningful. If not, explain why? If so, state whether it is a vector or a scalar:

$\left(\bar{a} . \bar{b}\right) \bar{c}$

Solution: This is a scalar multiplication of a vector. Therefore, this expression is meaningful and it is a **vector**.

Miscellaneous exercise 5 | Q 34.09 | Page 192

State whether the expression is meaningful. If not, explain why? If so, state whether it is a vector or a scalar:

$|\bar{a}| \left(\bar{b} . \bar{c}\right)$

Solution: This is the product of two scalars. Therefore, this expression is meaningful and it is a **scalar**.

Miscellaneous exercise 5 | Q 34.1 | Page 192

State whether the expression is meaningful. If not, explain why? If so, state whether it is a vector or a scalar:

$\bar{a}.\left(\bar{b}+\bar{c}\right)$

Solution: This is the scalar product of two vectors. Therefore, this expression is meaningful and it is a **scalar.**

Miscellaneous exercise 5 | Q 34.11 | Page 192

State whether the expression is meaningful. If not, explain why? If so, state whether it is a vector or a scalar:

$\bar{a}.\bar{b}+\bar{c}$

Solution: This is the sum of scalar and vector which is not defined. Therefore, this expression is meaningless.

Miscellaneous exercise 5 | Q 34.12 | Page 192

State whether the expression is meaningful. If not, explain why? If so, state whether it is a vector or a scalar:

$|\bar{a}|.\left(\bar{b}+\bar{c}\right)$

Solution:

This is meaningless because $\bar{a}$ is a vector, $\bar{b}+\bar{c}$ is a scalar and the scalar product of vector and scalar is not defined.

Miscellaneous exercise 5 | Q 35 | Page 192

For any vectors $\bar{a}, \bar{b}, \bar{c}$ show that

$\left(\bar{a}+\bar{b}+\bar{c}\right)\times\bar{c}+\left(\bar{a}+\bar{b}+\bar{c}\right)\times\bar{b}+\left(\bar{b}-\bar{c}\right)\times\bar{a}=2\bar{a}\times\bar{c}$

Solution:

LHS = $\left(\bar{a}+\bar{b}+\bar{c}\right)\times\bar{c}+\left(\bar{a}+\bar{b}+\bar{c}\right)\times\bar{b}+\left(\bar{b}-\bar{c}\right)\times\bar{a}$

$=\bar{a}\times\bar{c}+\bar{b}\times\bar{c}+\bar{c}\times\bar{c}+\bar{a}\times\bar{b}+\bar{b}\times\bar{b}+\bar{c}\times\bar{b}+\bar{b}\times\bar{a}-\bar{c}\times\bar{a}$

$=\bar{a}\times\bar{c}+\bar{b}\times\bar{c}+\bar{0}+\bar{a}\times\bar{b}+\bar{0}-\bar{b}\times\ \bar{c}-\bar{a}\times\ \bar{b}+\bar{a}\times\bar{c}......\left[\because\bar{a}\times\bar{b}=-\bar{b}\times\bar{a}\right]$

$=2\bar{a}\times\bar{c}$ = RHS.

Miscellaneous exercise 5 | Q 36.1 | Page 192

Suppose $\bar{a}=\bar{0}$:

If $\bar{a}.\bar{b}=\bar{a}.\bar{c}$, then is $\bar{b}=\bar{c}$?

Solution:

Take $\bar{b}=\hat{i},\bar{c}=\hat{j}$

Then $\bar{a}.\bar{b}=\bar{a}.\bar{c}=0$, but $\bar{b}\neq\bar{c}$

Miscellaneous exercise 5 | Q 36.2 | Page 192

Suppose $\bar{a}=\bar{0}$:

If $\bar{a}\times\bar{b}=\bar{a}\times\bar{c}$, then is $\bar{b}=\bar{c}$?

Solution:

Take $\bar{b}=\hat{i},\bar{c}=\hat{j}$

Then $\bar{a}\times\bar{b}=\bar{a}\times\bar{c}=0$, but $\bar{b}\neq\bar{c}$

Miscellaneous exercise 5 | Q 36.3 | Page 192

Suppose $\bar{a}=\bar{0}$:

If $\bar{a}.\bar{b}=\bar{a}.\bar{c}$ and $\bar{a}\times\bar{b}=\bar{a}\times\bar{c}$, then is $\bar{b}=\bar{c}$?

Solution:

Take $\bar{a} = \bar{0}, \bar{b} = \hat{i}, \bar{c} = \hat{j}$

Then $\bar{a}.\bar{b} = \bar{a}.\bar{c} = 0$ and $\bar{a} \times \bar{b} = \bar{a} \times \bar{c} = \bar{0}$, but $\bar{b} \neq \bar{c}$

Miscellaneous exercise 5 | Q 37.1 | Page 192

If A(3, 2, -1), B(-2, 2, -3), C(3, 5, -2), D(-2, 5, -4) then verify that the points are the vertices of a parallelogram.

Solution:

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ be the position vectors of A, B, C, D respectively w.r.t. the origin O.

Then $\bar{a} = 3\hat{i} + 2\hat{j} - \hat{k}, \bar{b} = -2\hat{i} + 2\hat{j} - 3\hat{k}, \bar{c} = 3\hat{i} + 5\hat{j} - 2\hat{k}, \bar{d} = -2\hat{i} + 5\hat{j} - 4\hat{k}.$

$\therefore \overline{AB} = \bar{b} - \bar{a}$

$= \left(-2\hat{i} + 2\hat{j} - 3\hat{k}\right) - \left(3\hat{i} + 5\hat{j} - \hat{k}\right)$

$= -5\hat{i} - 2\hat{k}$

$\therefore \overline{DC} = \bar{c} - \bar{d}$

$= \left(3\hat{i} + 5\hat{j} - 2\hat{k}\right) - \left(-2\hat{i} + 5\hat{j} - 4\hat{k}\right)$

$= 5\hat{i} + 2\hat{k}$

$= -\left(-5\hat{i} - 2\hat{k}\right)$

$\therefore \overline{DC} = -\overline{AB}$

$\therefore \overline{DC}$ is scalar multiple of $\overline{AB}$

$\therefore \overline{DC}$ is parallel to $\overline{AB}$

Also, $\left|\overline{DC}\right| = \sqrt{5^2 + 2^2} = \sqrt{25 + 4} = \sqrt{29}$

and $\left|\overline{AB}\right| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{25 + 4} = \sqrt{29}$

$\therefore \left|\overline{DC}\right| = \left|\overline{AB}\right|$

$\therefore$ l(AB) = l(DC)

$\therefore$ opposite sides AB and DC of ABCD are parallel and equal.

$\therefore$ ABCD is a parallelogram.

Miscellaneous exercise 5 | Q 37.2 | Page 192

If A(3, 2, -1), B(-2, 2, -3), C(3, 5, -2), D(-2, 5, -4) then find its area.

Solution:

$$\overline{AB} = \bar{b} - \bar{a} = \left(-2\hat{i} + 2\hat{j} - 3\hat{k}\right) - \left(3\hat{i} + 2\hat{j} - \hat{k}\right)$$

$$= -5\hat{i} - 2\hat{k}$$

$$\overline{AD} = \bar{d} - \bar{a} = \left(-2\hat{i} + 5\hat{j} - 4\hat{k}\right) - \left(3\hat{i} + 2\hat{j} - \hat{k}\right)$$

$$= -5\hat{i} + 3\hat{j} - 3\hat{k}$$

$$\therefore \overline{AB} \times \overline{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & 0 & -2 \\ -5 & 3 & -3 \end{vmatrix}$$

$$= (0 + 6)\hat{i} - (15 - 10)\hat{j} + (-15 + 0)\hat{k}$$

$$= 6\hat{i} - 5\hat{j} - 15\hat{k}$$

$\therefore$ area of parallelogram = $\left|\overline{AB} \times \overline{AD}\right|$

$$= \sqrt{6^2 + (-5)^2 + (-15)^2}$$

$$= \sqrt{36 + 25 + 225}$$

$= \sqrt{286}$ sq units.

Miscellaneous exercise 5 | Q 38 | Page 193

Let A, B, C, D be any four points in space. Prove that

$\left|\overline{AB} \times \overline{CD} + \overline{BC} \times \overline{AD} + \overline{CA} + \overline{BD}\right|$ = 4 (area of triangle ABC).

Solution:

Let A, B, C, D have position vectors $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ respectively.

Consider $\overline{AB} \times \overline{CD} + \overline{BC} \times \overline{AD} + \overline{CA} \times \overline{BD}$

$= (\bar{b} - \bar{a}) \times (\bar{d} - \bar{c}) + (\bar{c} - \bar{b}) \times (\bar{d} - \bar{a}) + (\bar{a} - \bar{c}) \times (\bar{d} - \bar{b})$

$= \bar{b} \times (\bar{d} - \bar{c}) - \bar{a} \times (\bar{d} - \bar{c}) + \bar{c} \times (\bar{d} - \bar{a}) - \bar{b} \times (\bar{d} - \bar{a}) + \bar{a} \times (\bar{d} - \bar{b}) - \bar{c} \times (\bar{d} - \bar{b})$

$= \bar{b} \times \bar{d} - \bar{b} \times \bar{c} - \bar{a} \times \bar{d} + \bar{a} \times \bar{c} + \bar{c} \times \bar{d} - \bar{c} \times \bar{a} - \bar{b} \times \bar{d} + \bar{b} \times \bar{a} + \bar{a} \times \bar{d} - \bar{a} \times \bar{b} - \bar{c} \times \bar{d} + \bar{c} \times \bar{b}$

$= \bar{b} \times \bar{d} - \bar{b} \times \bar{c} - \bar{a} \times \bar{d} - \bar{c} \times \bar{a} + \bar{c} \times \bar{d} - \bar{c} \times \bar{a} - \bar{b} \times \bar{d} - \bar{a} \times \bar{b} + \bar{a} \times \bar{d} - \bar{a} \times \bar{b} - \bar{c} \times \bar{d} - \bar{b} \times \bar{c}$ .... $[\because \bar{p} \times \bar{q} = -\bar{q} \times \bar{p}]$

$= -2(\bar{a} \times \bar{b} + \bar{b} \times \bar{c} + \bar{c} \times \bar{a})$

$\therefore \left|\overline{AB} \times \overline{CD} + \overline{BC} \times \overline{AD} + \overline{CA} \times \overline{BD}\right|$

= $\left|-2(\bar{a} \times \bar{b} + \bar{b} \times \bar{c} + \bar{c} \times \bar{a})\right|$

$= 4\left[\frac{1}{2}\left|\bar{a} \times \bar{b} + \bar{b} \times \bar{c} + \bar{c} \times \bar{a}\right|\right]$

= 4(are of Δ ABC).

Miscellaneous exercise 5 | Q 39 | Page 192

Let hat"a", hat"b", hat"c" be unit vectors such that hat"a".hat"b" = hat"a".hat"c" = 0 and 6 the angle between hat"b" and hat"c" is pi/6. Prove that hat"a" = +- 2(hat"b" xx hat"c").

Solution:

$\hat{a}.\hat{b} = \hat{a}.\hat{c} = 0$

$\therefore$ $\hat{a}$ is perpendicular to $\hat{b}$ and $\hat{c}$ both

$\therefore$ $\hat{a}$ is parallel to $\hat{b} \times \hat{c}$

$\therefore \hat{a} = m\left(\hat{b} \times \hat{c}\right)$, m is a scalar.

$\therefore |\hat{a}| = |m|\left|\hat{b} \times \hat{c}\right|$

$\therefore |\hat{a}| = |m|\left|\hat{b}\right||\hat{c}|$

$\therefore |\hat{a}| = |m|\left|\hat{b} \times \hat{c}\right| \frac{\sin \pi}{6}$

$\therefore 1 = |m| \times 1 \times 1 \times \frac{1}{2} = \frac{|m|}{2}$ ..... $\left[\because |\hat{a}| = \left|\hat{b}\right| = |\hat{c}| = 1\right]$

$\therefore 2 = |m|$

$\therefore m = \pm 2$

$\therefore \hat{a} = \pm 2\left(\hat{b} \times \hat{c}\right).$

Miscellaneous exercise 5 | Q 40 | Page 192

Find the value of ‘a’ so that the volume of parallelopiped formed by hat"i" + "a"hat"j" + hat"k", hat"j" + "a"hat"k" and "a"hat"i" + hat"k" becomes minimum.

Solution:

Let $\bar{p} = \hat{i} + a\hat{j} + \hat{k}, \bar{q} = \hat{j} + a\hat{k}, \bar{r} = a\hat{i} + \hat{k}$

Let V be the volume of the parallelopiped formed by $\bar{p}, \bar{q}, \bar{r}$.

Then V = $[\bar{p}\bar{q}\bar{r}]$

$$= \begin{vmatrix} 1 & a & 1 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix}$$

$= 1(1-0) - a\left(0 - a^2\right) + 1(0 - a)$

$= 1 + a^3 - a$

$\therefore \frac{dV}{da} = \frac{d}{da}\left(1 + a^3 - a\right)$

$= 0 + 3a^2 - 1 = 3a^2 - 1$

and $\frac{d^2V}{da^2} = \frac{d}{da}\left(3a^2 - 1\right)$

$= 3 \times 2a - 0 = 6a$

For maximum and minimum V, $\frac{dV}{da} = 0$

$\therefore 3a^2 - 1 = 0$

$\therefore a^2 = \frac{1}{3}$

$\therefore a = \pm\frac{1}{\sqrt{3}}$

Now, $\left(\frac{d^2V}{da^2}\right)_{\text{at } a=\frac{1}{\sqrt{3}}} = 6\left(\frac{1}{\sqrt{3}}\right) = 2\sqrt{3} > 0$

$\therefore$ V is minimum when a = $\frac{1}{\sqrt{3}}$

Also, $\left(\frac{d^2V}{da^2}\right)_{\text{at } a=-\frac{1}{\sqrt{3}}} = 6\left(-\frac{1}{\sqrt{3}}\right) = -2\sqrt{3} < 0$

$\therefore$ V is minimum when a = $-\frac{1}{\sqrt{3}}$

Hence, a = $\frac{1}{\sqrt{3}}$

Miscellaneous exercise 5 | Q 41 | Page 193

Find the volume of the parallelopiped spanned by the diagonals of the three faces of a cube of side a that meet at one vertex of the cube.

Solution: Take origin O as one vertex of the cube and OA, OB and OC as the positive directions of the X-axis, the Y-axis and the Z-axis respectively. Here, the sides of the cube are
OA = OB = OC = a

$\therefore$ the coordinates of all the vertices of the cube will be

O (0, 0, 0) B(0, a, 0) N(a, a, 0) M(a, 0, a) A(a, 0, 0) C(0, 0, a) L (0, a, a) P(a, a, a)

ON, OL, OM are the three diagonals which meet at the vertex O

$\overline{ON} = \bar{a}\hat{i} + \bar{a}\hat{j}, \overline{OL} = \bar{a}\hat{j} + \bar{a}\hat{k}$

$\overline{OM} = \bar{a}\hat{i} + \bar{a}\hat{k}$

$$\left[\overline{ON}\ \overline{OL}\ \overline{OM}\right] = \begin{vmatrix} a & a & 0 \\ 0 & a & a \\ a & 0 & a \end{vmatrix}$$

= $a(a^2 - 0) - a(0 - a^2) + 0$

= $a^3 + a^3 = 2a^3$

∴ required volume = $\left[\overline{ON}\overline{OL}\overline{OM}\right]$

= $2a^3$ cubic units

Miscellaneous exercise 5 | Q 42 | Page 192

If $\bar{a}, \bar{b}, \bar{c}$ are three non-coplanar vectors show that $\dfrac{\bar{a}.\left(\bar{b} \times \bar{c}\right)}{(\bar{c} \times \bar{a}).\bar{b}} + \dfrac{\bar{b}.(\bar{a} \times \bar{c})}{(\bar{c} \times \bar{a}).\bar{b}} = 0$

Solution:

LHS = $\dfrac{\bar{a}.\left(\bar{b} \times \bar{c}\right)}{(\bar{c} \times \bar{a}).\bar{b}} + \dfrac{\bar{b}.(\bar{a} \times \bar{c})}{(\bar{c} \times \bar{a}).\bar{b}}$

$= \dfrac{\left(\bar{a}\bar{b}\bar{c}\right)}{\left(\bar{c}\bar{a}\bar{b}\right)} + \dfrac{\left(\bar{b}\bar{a}\bar{c}\right)}{\left(\bar{c}\bar{a}\bar{b}\right)}$

$= \dfrac{\left(\bar{a}\ \bar{b}\ \bar{c}\right)}{\left(\bar{a}\ \bar{b}\ \bar{c}\right)} - \dfrac{\left(\bar{a}\ \bar{b}\ \bar{c}\right)}{\left(\bar{a}\ \bar{b}\ \bar{c}\right)}$

= 0 = RHS.

Miscellaneous exercise 5 | Q 43 | Page 193

Prove that $\left(\bar{a} \times \bar{b}\right).\left(\bar{c} \times \bar{d}\right)$ = $\begin{vmatrix} \bar{a}.\bar{c} & \bar{b}.\bar{c} \\ \bar{a}.\bar{d} & \bar{b}.\bar{d} \end{vmatrix}$.

Solution:

Let $\bar{a} \times \bar{b} = \bar{m}$

LHS = $\left(\bar{a} \times \bar{b}\right).\left(\bar{c}.\bar{d}\right)$

$= \bar{m}.\left(\bar{c} \times \bar{d}\right)$

$= \left(\bar{m} \times \bar{c}\right).\bar{d}$

$= \left[\left(\bar{a} \times \bar{b}\right) \times \bar{c}\right].\bar{d}$

$= \left[(\bar{c}.\bar{a})\bar{b} - \left(\bar{c}.\bar{b}\right)\bar{a}\right].\bar{d}$

$= \left(\bar{c}.\bar{a}\right)\left(\bar{b}.\bar{d}\right) - \left(\bar{c}.\bar{b}\right)\left(\bar{a}.\bar{d}\right)$

= $\begin{vmatrix} \bar{c}.\bar{a} & \bar{c}.\bar{b} \\ \bar{a}.\bar{d} & \bar{b}.\bar{d} \end{vmatrix}$.

= $\begin{vmatrix} \bar{a}.\bar{c} & \bar{b}.\bar{c} \\ \bar{a}.\bar{d} & \bar{b}.\bar{d} \end{vmatrix}$. .....[Do product is commutative]

= RHS.

Miscellaneous exercise 5 | Q 44 | Page 193

Find the volume of a parallelopiped whose coterimus edges are represented by the vectors $\hat{i} + \hat{k}, \hat{i} + \hat{k}, \hat{i} + \hat{j}$. Also find volume of tetrahedron having these coterminus edges.

Solution:

Let $\bar{a} = \hat{j} + \hat{k}, \bar{b} = \hat{i} + \hat{k}$ and $\bar{c} = \hat{i} + \hat{j}$ be the coterminus edges of a parallelopiped.

Then volume of the parallelopiped = $[\bar{a}\ \bar{b}\ \bar{c}]$

$$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

= 0(0 - 1)- 1(0 - 1) + 1(1 - 0)

= 0 + 1 + 1 = 2 cu units

Also, volume of tetrahedron = $\frac{1}{6}[\bar{a}\ \bar{b}\ \bar{c}]$

$= \frac{1}{6}(2) = \frac{1}{3}$ cubic units

Miscellaneous exercise 5 | Q 45 | Page 193

Using properties of scalar triple product, prove that $[\bar{a} + \bar{b}\ \bar{b} + \bar{c}\ \bar{c} + \bar{a}] = 2[\bar{a}\ \bar{b}\ \bar{c}]$.

Solution:

LHS = $[\bar{a} + \bar{b}\ \ \bar{b} + \bar{c}\ \ \bar{c} + \bar{a}]$

$= (\bar{a} + \bar{b}).\{(\bar{b} + \bar{c}) \times (\bar{c} + \bar{a})\}$

$= (\bar{a} + \bar{b}).\{\bar{b} \times \bar{c} + \bar{b} \times \bar{a} + \bar{c} \times \bar{c} + \bar{c} \times \bar{a}\}$

$= (\bar{a} + \bar{b}).(\bar{b} \times \bar{c} + \bar{b} \times \bar{a} + \bar{c} \times \bar{a})$ .... $[\because \bar{c} \times \bar{c} = \bar{0}]$

$= \bar{a}.\{(\bar{b} \times \bar{c}) + (\bar{b} \times \bar{a}) + (\bar{c} \times \bar{a})\} + \bar{b}.\{(\bar{b} \times \bar{c}) + (\bar{b} \times \bar{a}) + (\bar{c} \times \bar{a})\}$

$= \bar{a}.(\bar{b} \times \bar{c}) + \bar{a}.(\bar{b} \times \bar{a}) + \bar{a}.(\bar{c} \times \bar{a}) + \bar{b}.(\bar{b} \times \bar{c}) + \bar{b}(\bar{b} \times \bar{a}) + \bar{b}(\bar{c} \times \bar{a})$

$= [\bar{a}\ \bar{b}\ \bar{c}] + [\bar{a}\ \bar{b}\ \bar{a}] + [\bar{a}\ \bar{c}\ \bar{a}] + [\bar{b}\ \bar{b}\ \bar{c}] + [\bar{b}\ \bar{b}\ \bar{a}] + [\bar{b}\ \bar{c}\ \bar{a}]$

$= [\bar{a}\ \bar{b}\ \bar{c}] + 0 + 0 + 0 + 0 + [\bar{a}\ \bar{b}\ \bar{c}]$

$= 2[\bar{a}\ \bar{b}\ \bar{c}]$

= RHS

Miscellaneous exercise 5 | Q 46 | Page 193

If four points $A(\bar{a}), B(\bar{b}), C(\bar{c})$ and $D(\bar{d})$ are coplanar, then show that $[\bar{a}\bar{b}\bar{c}] + [\bar{b}\bar{c}\bar{d}] + [\bar{c}\bar{a}\bar{d}] = [\bar{a}\bar{b}\bar{c}]$.

Solution:

$\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$ are the position vectors of the points A, B, C and D respectively.

$\therefore \overline{AB} = \bar{b} - \bar{a}, \overline{AC} = \bar{c} - \bar{a}, \overline{AD} = \bar{d} - \bar{a}$

The points A, B, C, D are coplanar.

$\therefore$ the vectors $\overline{AB}, \overline{AC}, \overline{AD}$ are coplanar.

$\therefore \left[\overline{AB}\ \overline{AC}\ \overline{AD}\right] = 0$

$\therefore [\bar{b} - \bar{a}\ \ \bar{c} - \bar{a}\ \ \bar{d} - \bar{a}] = 0$

$\therefore (\bar{b} - \bar{a}).[(\bar{c} - \bar{a}) \times (\bar{d} - \bar{a})] = 0$

$\therefore (\bar{b} - \bar{a}).(\bar{c} \times \bar{d} - \bar{c} \times \bar{a} - \bar{a} \times \bar{d} + \bar{a} \times \bar{a}) = 0\ where\ \bar{a} \times \bar{a} = \bar{0}$

$\therefore (\bar{b} - \bar{a}).(\bar{c} \times \bar{d} - \bar{c} \times \bar{a} - \bar{a} \times \bar{d}) = 0$

$\therefore \bar{b}.(\bar{c} \times \bar{d}) - \bar{b}.(\bar{c} \times \bar{a}) - \bar{b}.(\bar{a} \times \bar{d}) - \bar{a}.(\bar{c} \times \bar{d}) + \bar{a}.(\bar{c} \times \bar{a}) + \bar{a}.(\bar{a} \times \bar{d}) = 0$ .....(1)

$\therefore$ Now, $\bar{a}.(\bar{c} \times \bar{a}) = 0, \bar{a}.(\bar{a} \times \bar{d}) = 0,$

$-\bar{b}.(\bar{a} \times \bar{d}) = \bar{b}.(\bar{d} \times \bar{a}) = [\bar{b}\bar{d}\bar{a}] = [\bar{a}\bar{b}\bar{d}],$

$-\bar{a}.(\bar{c} \times \bar{d}) = \bar{a}.(\bar{d} \times \bar{c}) = [\bar{a}\bar{d}\bar{c}] = [\bar{c}\bar{a}\bar{d}]$

Also, $\bar{b}.(\bar{c} \times \bar{d}) = [\bar{b}\bar{c}\bar{d}],$

$-\bar{b}.(\bar{c} \times \bar{a}) = -\bar{a}.(\bar{b} \times \bar{c}) = -[\bar{a}\bar{b}\bar{c}]$

$\therefore$ from (1)

$[\bar{b}\bar{c}\bar{d}] - [\bar{a}\bar{b}\bar{c}] + [\bar{a}\bar{b}\bar{d}] + [\bar{c}\bar{a}\bar{d}] + 0 + 0 = 0$

$\therefore [\bar{a}\bar{b}\bar{d}] + [\bar{b}\bar{c}\bar{d}] + [\bar{c}\bar{a}\bar{d}] = [\bar{a}\bar{b}\bar{c}]$

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If a¯,b¯,c¯ are three non-coplanar vectors, then (a¯+b¯+c¯).[(a¯+b¯)×(a¯+c¯)]=-[a¯ b¯c¯]

Solution:

$$\text{LHS} = \left(\bar{a} + \bar{b} + \bar{c}\right) . \left[\left(\bar{a} + \bar{b}\right) \times \left(\bar{a} + \bar{c}\right)\right]$$

$$= \left(\bar{a} + \bar{b} + \bar{c}\right) . \left[\bar{a} \times \left(\bar{a} + \bar{c}\right) + \bar{b}\left(\bar{a} \times \bar{c}\right)\right]$$

$$= \left(\bar{a} + \bar{b} + \bar{c}\right)\left[\bar{a} \times \bar{a} + \bar{a} \times \bar{c} + \bar{b} \times \bar{a} + \bar{b} \times \bar{c}\right]$$

$$= \bar{a}.\left(\bar{a} \times \bar{c}\right) + \bar{a}.\left(\bar{b} \times \bar{a}\right) \times \bar{a}.\left(\bar{b} \times \bar{c}\right) + \bar{b}.\left(\bar{a} \times \bar{c}\right) + \bar{b}.\left(\bar{b} \times \bar{a}\right) + \bar{b}.\left(\bar{b} \times \bar{c}\right) + \bar{c}.\left(\bar{a} \times \bar{c}\right) + \bar{c}.\left(\bar{b} \times \bar{a}\right) \times \bar{c}.\left(\bar{b} \times \bar{c}\right) \quad \ldots\ldots. \left[\because \bar{a} \times \bar{a} = \bar{0}\right]$$

$$= \left[\bar{a}\ \bar{a}\ \bar{c}\right] + \left[\bar{a}\ \bar{b}\ \bar{a}\right] + \left[\bar{a}\bar{b}\bar{c}\right] + \left[\bar{b}\bar{a}\bar{c}\right] + \left[\bar{b}\ \bar{b}\ \bar{a}\right] + \left[\bar{b}\ \bar{b}\ \bar{c}\right] + \left[\bar{c}\ \bar{a}\ \bar{c}\right] + \left[\bar{c}\ \bar{b}\ \bar{a}\right] + \left[\bar{c}\ \bar{b}\ \bar{c}\right]$$

$$= 0 + 0 + \left[\bar{a}\bar{b}\bar{c}\right] - \left[\bar{a}\bar{b}\bar{c}\right] + 0 + 0 + 0 - \left[\bar{a}\bar{b}\bar{c}\right] + 0$$

$$= -\left[\bar{a}\bar{b}\bar{c}\right] = \text{RHS.}$$

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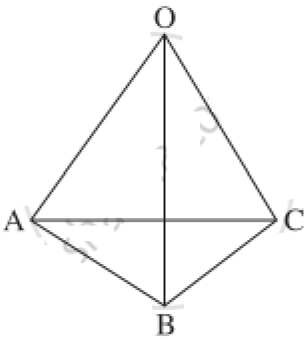
If in a tetrahedron, edges in each of the two pairs of opposite edges are perpendicular, then show that the edges in the third pair is also perpendicular.

Solution:



Let O-ABC be a tetrahedron. Then (OA, BC), (OB, CA) and (OC, AB) are the pair of opposite edges.

Take O as the origin of reference and let $\bar{a}, \bar{b}$ and $\bar{c}$ be the position vectors of the vertices A, B and C respectively. Then

$\overline{OA} = \bar{a},\ \overline{OB} = \bar{b},\ \overline{OC} = \bar{c},$

$\overline{AB} = \bar{b} - \bar{a}, \overline{BC} = \bar{c} - \bar{b}$ and $\overline{CA} = \bar{a} - \bar{c}$

Now, suppose the pairs (OA, BC) and (OB, CA) are perpendicular to each other.

Then $\overline{OA}.\overline{BC} = 0, i.e. \bar{a}.\left(\bar{c} - \bar{b}\right) = 0$

$\therefore \bar{a}.\bar{c} - \bar{a}.\bar{b} = 0$ ....(1)

and $\overline{OB}.\overline{CA} = 0, i.e. \bar{b}.(\bar{a} - \bar{c}) = 0$

$\therefore \bar{b}.\bar{a} - \bar{b}.\bar{c} = 0$

$\therefore \bar{a}.\bar{b} - \bar{b}.\bar{c} = 0$ ...(2)

Adding (1) and (2), we get

$\bar{a}.\bar{c} - \bar{b}.\bar{c} = 0$

$\therefore \bar{c}.\bar{b} - \bar{c}.\bar{a} = 0$

i.e. $\bar{c}.\left(\bar{b} - \bar{a}\right) = 0$

$\therefore\ \overline{OC}.\overline{AB} = 0$

$\therefore$ the third pair (OC, AB) is perpendicular.